[Wuhu, July 24, 2008]

Exit Measure of Super-diffusions in Random Medium

Yan-Xia Ren

(Peking University)

(Based on a joint work with Panki Kim, Renming Song and Hao Wang)

Outline:

1) Introduction of the exit measure of super-diffusions

2) Superdiffusions in random medium

3) Absolute continuity of exit measure X_D

4) Regularity of the density of exit measure X_D

1. Introduction of the exit measure of Super-diffusions

1) Intuitive definition (Branching Particle Systems)

ξ: A diffusion in R^d with generator L.

Consider a system of particles with undergo random motion and branching on R^d according to the following rules:

a). At time $t = 0$, we have finite number of particles which are distributed according to the law of the Poisson point process on R^d with intensity $n\mu$. (Each particle has mass $1/n$)

b). Each particle survives with probability $\exp(-nt)$ at time t.

c). At the end of its lifetime, a dying particle gives birth to k offsprings with probability $p_k^n, k = 0, 1, 2, \cdots$.

d). During its lifetime, the motion of each particle is governed by the process ξ.

e). All particle's lifetime, motions, and branching are independent of one another.

For a Borel set $B \in \mathrm{R}^d$, put

$$X_t^n(B) = \frac{1}{n}(\#\text{ of particles which are alive at time } t \text{ and located in } B)$$

Suppose D is a domain. For Borel set $B \in \mathcal{B}(\partial D)$, put

$$X_D^n(B) = \frac{1}{n}(\#\text{ of particles whose exit point from } D \text{ is in } B).$$

Under certain conditions (**see remark below**), it can be proved that as $n \to \infty$,

$$X_t^n \Longrightarrow X_t \quad X_D^n \Longrightarrow X_D \quad \text{(weakly)}.$$

We call $X = \{X_t, X_D; P_\mu\}$ a super-diffusion (enhanced model).

X_t describes the mass distribution of particles at time t.

X_D is called the exit measure from D, which describes the mass distribution of exit points.

Remark Main conditions on $p^n = (p_k^n, k = 0, 1, 2, \cdots)$:

$$p_k^{(n)} = 0 \qquad \text{if } k = 1 \text{ or } k \geq n + 1,$$

and

$$\sum_{k=0}^{n} k p_k^{(n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{k\geq 0} |p_k^{(n)} - p_k| = 0,$$

where $\{p_k, k = 0, 1, 2, \cdots\}$ is the limiting offspring distribution which is assumed to satisfy following conditions:

$$p_1 = 0, \sum_{k=0}^{\infty} k p_k = 1 \text{ and } m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

2) (Dynkin's definition)The Laplace functional of X_D is given by

$$P_\mu \exp(-\langle f, X_D\rangle) = \exp(-\langle u, \mu\rangle), f \in b\mathcal{B}(\mathrm{R}^d),$$

where u is the unique solution to the following integral equation:

$$u(x) + \int_0^{\tau_D} u^2(\xi_s)ds = \Pi_x \left[f(\xi_{\tau_D})\right]. \tag{1}$$

Differential form:

$$\begin{cases} Lu = u^2, \quad \text{in } D, \\ u|_{\partial D} = f(x). \end{cases}$$

But this method does not work for super-diffusions in a random medium due to interaction between particles, which destroys the multiplicative property:

$$\begin{aligned} & P_{\mu_1+\mu_2} \exp(-\langle f, X_D \rangle) \\ = \quad & P_{\mu_1} \exp(-\langle f, X_D \rangle) \cdot P_{\mu_2} \exp(-\langle f, X_D \rangle), \end{aligned}$$

which can be formally written as

$$(X_D, P_{\mu_1+\mu_2}) = (X_D, P_{\mu_1}) + (X_D, P_{\mu_2}) \quad \text{in law} .$$

It is well-known that the log-Laplace functional technique is based on the multiplicative property.

Question: How to define exit measure X_D for super-diffusions in random medium.

3) (Le Gall and Mytnik's definition) In Le Gall and Mytnik (2005), the exit measure of a super-Brwonian motion was obtained by an approximation method.

$B = (B_t, \Pi_x)$: A Brownian motion in $\mathbb{R}^d$ starting from x.

$K_D(x, z)$: the Poisson kernel of Brownian motion B in D;
$G_D(x, y)$: the Green function of Brownian motion B in D.

A fact: $K_D(x, z)$ is half the normal derivative of the mapping $y \longrightarrow G_D(x, y)$ at $z \in \partial D$, in other words,

$$G_D(x, y) \sim 2\rho(y) K_D(x, z), \quad y \longrightarrow z \text{ along the normal direction.}$$

BM	Super-BM

BM

δ_{B_t}

$\delta_{B_{\tau_D}}$(exit measure) :

$$\Pi_x\langle\phi, \delta_{B_{\tau_D}}\rangle$$

$$= \int_D K_D(x,z)\phi(z)\sigma(dz)$$

Occupation time measure:

$$O_D := \int_0^{\tau_D} \delta_{B_t} dt = \int_0^\infty \delta_{\xi_t} dt$$

$$\Pi_x\langle\phi, O_D\rangle = \int_D G_D(x,y)\phi(y)dy$$

Super-BM

X_t

X_D(exit measure) :

How to describe

$\langle\phi, X_D\rangle$?

$$Y_D := \int_0^\infty X_t^D dt$$

Y_D can be described

Here

$$\xi_t := \begin{cases} B_t, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

X_t^D is the super-Breonian motion in D (with underling motion ξ).

For every $\epsilon > 0$ set

$$F_\epsilon = \{x \in D; \rho(x, \partial D) \leq \epsilon\}.$$

Then we have

$$\begin{aligned} \frac{1}{\epsilon^2}\Pi_x\left[\int_0^\infty I_{F_\epsilon}(\xi_t)\phi(\xi_t)dt\right] &= \frac{1}{\epsilon^2}\int_{F_\epsilon} G_D(x,y)\phi(y)dy \\ &\to \int_{\partial D} K_D(x,z)\phi(z)\sigma(dz). \end{aligned}$$

Suppose $X = (X_t, t \geq 0)$ is a super-Brownian motion in D. More precisely, the underlying spatial motion ξ is a Brownian motion killed when it exits D. It is reasonable to have

$$X_D^\epsilon(dy) := \frac{1}{\epsilon^2}\int_0^\infty I_{F_\epsilon}(y)X_t(dy)dt \Longrightarrow X_D(\text{ as } \epsilon \to 0).$$

2. Super-diffusion in random medium (Ren, song and Wang 08)

Suppose that for each $k \in \mathbb{N}$, z_k is the strong solution of the following equation:

$$\begin{aligned} z_k(t) - z_k(0) = & \int_0^t c(z_k(s)) dB_k(s) \\ & + \int_0^t \int_{\mathbb{R}^d} h(y - z_k(s)) W(dy, ds), \quad (2) \end{aligned}$$

where $\{B_k, k \geq 1\}$ are independent $\mathbb{R}^d$-valued, standard Brownian motions, W is a Brownian sheet or space-time white noise on $\mathbb{R}^d$.

Assume that the diffusion matrix $(a_{pq})_{1\leq p,q\leq d}$ defined by

$$a_{pq}(x) := \sum_{r=1}^{d} c_{pr}(x)c_{qr}(x), \tag{3}$$

is uniformly elliptic and bounded on $\mathbb{R}^d$. Assume further that $h = (h_1, \cdots, h_d) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Put

$$\rho_{pq}(x, y) := \int_{\mathbb{R}^d} h_p(u - x)h_q(u - y)du, \quad p, q = 1, \cdots, d. \tag{4}$$

Notations:

$(\mathscr{D}(A), A)$: Generator of z_1 killed when it leaves D(denoted as ξ).

For $\phi \in \mathscr{D}(A)$,

$$A\phi(x) = \sum_{p,q=1}^{d} \frac{1}{2}(a_{pq}(x) + \rho_{pq}(0,0))\partial_p\partial_q\phi(x), \quad x \in D. \quad (5)$$

$M_F(D)$: the set of all finite measures on D;

$M_{F,c}(D) := \{\nu \in M_F(D) : \text{supp}(\nu) \subset D\}$.

$K_D(x, z)$: the Poisson kernel of ξ in D;

$G_D(x, y)$: the Green function of ξ in D.

For any measure $\mu \in M_{F,c}(D)$, there is a unique solution $X = (X_t, t \geq 0)$ (called super-diffusion in random medium) to the martingale problem:

$$\begin{aligned} & X_t(\phi) - X_0(\phi) \\ = \ & \sum_{p=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot)\partial_p \phi(\cdot), X_s \rangle W(dy, ds) \\ & + \int_0^t \int_D \phi(x) M(dx, ds) \\ & + \int_0^t \left\langle \sum_{p,q=1}^{d} \frac{1}{2}(a_{pq}(\cdot) + \rho_{pq}(\cdot,\cdot))\partial_p \partial_q \phi(\cdot), \ X_s \right\rangle ds \end{aligned} \quad (6)$$

for every $t > 0$ and $\phi \in C_c^\infty(D)$, where M is a square-integrable martingale measure with

$$\langle M(\phi)\rangle_t = \gamma\sigma^2 \int_0^t \langle \phi^2, X_u\rangle du \qquad \text{for every } t > 0 \text{ and } \phi \in C_c^\infty(D$$

Here

$$M_t(\phi) := \int_0^t \int_D \phi(y) M(ds, dy) \tag{7}$$

is a square-integrable, continuous $\{\mathcal{F}_t\}$-martingale, and

$$W_t(\phi) := \sum_{p=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot)\partial_p\phi(\cdot), X_s\rangle W(dy, ds),$$

$$\mathcal{F}_t := \sigma\{X_s(f), M_s(f), W_s(\phi), f \in \mathcal{B}(D), \phi \in C^1(D), s \le t\}.$$

Moreover, $W_t(\phi)$ and $M_t(\phi)$ are orthogonal.

3. Absolute continuity of the exit measure X_D

Recall that

$$F_\epsilon = \{x \in D; \rho(x, \partial D) \le \epsilon\}.$$

$$X_D^\epsilon(dy) := \frac{1}{\epsilon^2} \int_0^\infty I_{F_\epsilon}(y) X_t(dy) dt.$$

The following theorem provides a stochastic integral representation for the exit measure of a super-diffusion in random medium and its density when it exists.

Theorem 1 *(i) Then for every $\varphi \in C(\overline{D})$,*

$$\langle \varphi, X_D^\epsilon \rangle \to \langle \varphi, X_D \rangle \quad \mathbb{P}_\mu - \textit{a.s. and in } L^2(\Omega, \mathbb{P}_\mu),$$

where

$$\begin{aligned} & \langle \phi, X_D \rangle \\ = \; & \langle H_D \phi, \mu \rangle + \int_0^\infty \int_D H_D \phi(x) dM(s,x) \\ & + \sum_{p=1}^d \int_0^\infty \int_{\mathbb{R}^d} \langle h_p(z - \cdot) \partial_p (H_D \phi), X_s \rangle W(dz, ds). \end{aligned} \tag{8}$$

(ii) Suppose $d < 3$*. For every* $y \in \partial D$*, define*

$$\begin{aligned} & x_D(y) \\ = \; & \langle K_D(\cdot, y), \mu\rangle + \int_0^\infty \int_D K_D(x,y) dM(s,x) \\ & \sum_{p=1}^{d} \int_0^\infty \int_{\mathbb{R}^d} \langle h_p(z-\cdot)\partial_p(K_D(\cdot,y)), X_s\rangle W(dz, ds). \end{aligned} \tag{9}$$

Then, $x_D(y) \geq 0$*,* $\mathbb{P}_\mu$*-a.s. for every* $y \in \partial D$*. Finally,*

$$X_D(dy) = x_D(y)\sigma(dy), \quad \mathbb{P}_\mu - a.s.$$

where σ *is the surface measure on* ∂D *normalized to have total measure 1.*

4. Regularity of the density of exit measure

Note that, by definition (4), $\rho_{pq}(x,x) = \rho_{pq}(0,0)$. In this section we assume that $(a_{pq}(x))_{1\le p,q\le d}$ does not depends on spacial position x, then

$$A\phi(x) = \sum_{p,q=1}^{d} \frac{1}{2}(a_{pq} + \rho_{pq}(0,0))\partial_p\partial_q\phi(x), \quad x \in D, \tag{10}$$

where $(a_{pq})_{1\le p,q\le d}$ is a constant and positive definite matrix.

The next result gives the regularity of the density of X_D in a random medium.

Theorem 2 *Assume that* $d = 2$ *and the underling motion is a diffusion with constant diffusion matrix. For* $\mu \in M_{F,c}(D)$*, the processes* $(x_D(y), y \in \partial D)$ *under* P_μ *has a continuous version.*

Open question: If $(a_{pq}(x))_{1 \leq p,q \leq d}$ does depend on spacial position x, does the density x_D has continuous version?

Put

$$x_D^1(y) = \int_0^\infty \int_D K_D(x,y)dM(s,x),$$

and

$$x_D^2(y) = \sum_{p=1}^{d} \int_0^\infty \int_{\mathbb{R}^d} \langle \partial_p (K_D(\cdot, y)) h_p(z-\cdot), X_s \rangle W(dz, ds).$$

To prove that $(x_D(y), y \in \partial D)$ under P_μ has a continuous version we only need to prove that $(x_D^1(y), y \in \partial D)$ and $(x_D^2(y), y \in \partial D)$ under P_μ have continuous versions separately.

Lemma 3 *There exists a constant C such that, for $y_1, y_2 \in \partial D$,*

$$\mathbb{P}_\mu((x_D^1(y_1) - x_D^1(y_2))^4) \le C|y_1 - y_2|^2.$$

The processes $(x_D^1(y), y \in \partial D)$ has a continuous version.

Proof:

$$K_{y_1,y_2}(\cdot) := K_D(\cdot, y_1) - K_D(\cdot, y_2).$$

Then

$$x_D^1(y_1) - x_D^1(y_2) = \int_0^\infty \int_D K_{y_1,y_2}(x) dM(s, x). \qquad (11)$$

By the Burkholder-Davis-Gundy inequality

$$\mathbb{P}_\mu((x_D^1(y_1) - x_D^1(y_2))^4) \le C\mathbb{P}_\mu\left(\int_0^\infty \langle K^2_{y_1,y_2}, X_s\rangle ds\right)^2.$$

Note that

$$
\begin{aligned}
& \mathbb{P}_{\mu}\left(\int_0^\infty \langle K^2_{y_1,y_2}, X_s\rangle ds\right)^2 \\
\le \quad & C \sup_{x\in \text{supp}(\mu)} G_D(K_D(\cdot,y_1)-K_D(\cdot,y_2))^2(x) \\
& \cdot \sup_{x\in D} G_D\left[G_D(K_D(\cdot,y_1)-K_D(\cdot,y_2))^2\right](x) \\
\le \quad & C\,|y_1-y_2|\cdot|y_1-y_2|.
\end{aligned}
$$

Lemma 4 *There exists a constant C such that, for $y_1, y_2 \in \partial D$,*

$$\mathbb{P}_{\mu}((x_D^2(y_1) - x_D^2(y_2))^4) \leq C|y_1 - y_2|^2.$$

The processes $(x_D^2(y), y \in \partial D)$ has a continuous version.

Proof: By the Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
& \mathbb{P}_\mu((x_D^2(y_1) - x_D^2(y_2))^4) \\
\leq\ & C\mathbb{P}_\mu \left(\sum_{p=1}^{d} \int_0^\infty \int_{\mathbb{R}^d} \langle \partial_p (K_{y_1,y_2}) h_p(z-\cdot), X_s \rangle^2 ds dz) \right)^2 \\
\leq\ & \cdots \\
\leq\ & C|y_1 - y_2|^2 + \\
& C|y_1 - y_2| \sup_{x \in \mathrm{supp}(\mu)} |G_D \partial_i (K_D(\cdot, y_1) - K_D(\cdot, y_2))(x)| \\
& \cdot \sup_{x \in D} G_D \left| G_D \partial_i (K_D(\cdot, y_1) - K_D(\cdot, y_2)) \right| (x) \\
& + C|y_1 - y_2| \sup_{x \in D} G_D \left[G_D \partial_i (K_D(\cdot, y_1) - K_D(\cdot, y_2)) \right]^2 (x).
\end{aligned}
$$

Lemma 5 *Assume that $d = 2$ and A has constant coefficients. For any compact subset K of D and $i = 1, 2$, we have, for $z_1, z_2 \in \partial D$,*

$$\sup_{x \in K} |G_D \partial_i (K_D(\cdot, z_1) - K_D(\cdot, z_2))(x)| \leq C|z_1 - z_2|^{1/2};$$

$$\sup_{x \in D} G_D \left| G_D \partial_i (K_D(\cdot, z_1) - K_D(\cdot, z_2)) \right| (x) \leq C|z_1 - z_2|^{1/2};$$

and

$$\sup_{x \in D} G_D \left[G_D \partial_i (K_D(\cdot, z_1) - K_D(\cdot, z_2)) \right]^2 (x) \leq C|z_1 - z_2|.$$

D. A. Dawson, Z. Li and H. Wang (2001): *Superprocesses with dependent spatial motion and general branching densities,* Electron. J. Probab. **6(25)**, 1–33.

E. B. Dynkin (1991), *A probabilistic approach to one class of nonlinear differential equations*, Probab. Th. Rel. Fields, **89**, 89-115.

E. B. Dynkin (1993), *Superprocesses and partial differential equations*, Ann. Probab., **21**: 1185-1262.

J. F. Le Gall and L. Mytnik (2005), *Stochastic integral representation and regularity of the density for exit measure of super-Brownian motion*, Ann. Probab. **33(1)**, 194–222.

E. Perkins (2002), Dawson-watanabe superprocesses and measure-valued diffusions. *Lect. Notes Math.,* **1781**, 135-192, Springer-Verlag, Herdelberg.

Y.-X. Ren, R. Song, and H. Wang (2008), *A Class of Stochastic Partial Differential Equations for Interacting Superprocesses on a Bounded Domain*, To appear Osaka J. Math.

H. Wang (1997): *State classification for a class of measure-valued branching diffusions in a Brownian medium,* Probab. Th. Rel. Fields **109**, 39–55.

H. Wang (1998): *A class of measure-valued branching diffusions in a random medium,* Stochastic Anal. Appl. **16(4)**, 753–786.

Thank you!

E-mail: yxren@math.pku.edu.cn