Central Limit Theorem for Empirical Process under New Dependent Coefficient

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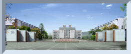

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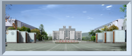


1 Introduction

➢Time series and random fields are main topics in modern statistical techniques. This is because they are essential for applications where randomness plays an important role. To describe the asymptotic behavior of certain time series or random fields,many ways of modeling the weak dependence have already been work out.

➢One of the most popular is the notion of mixing,there exists a wide literature on limit theorems under various classical mixing conditions such as strong mixing condition($\alpha-$mixing),absolute regularity ($\beta-$mixing), or $\varphi-$mixing. Examples for such conditions to hold are investigated in Doukhan(1994) and Rio(2000). However, many commonly used models for real-world phenomena do not satisfy classical mixing conditions. Moreover, a main inconvenience of mixing assumptions is the difficulty of checking them because the calculation involves the complicated manipulation of taking the supremum over two $\sigma-$ algebras.

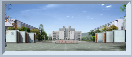
➢A reasonable question is then: how to modify the definition of the usual mixing coefficients in order to catch many more examples, without losing too much of their nice properties?

➢In this paper, we propose a new dependence measure($\gamma-$weak dependence) under the wasserstein metric for time series.

I) This definition make explicit the 'asymptotic independence' in some sense between 'past' and 'future' , some examples show that this weak dependence holds in many cases of interest.

II) Our condition is much easier to compute than mixing conditions for some financial time series models, moreover, we can give the computable rate of convergence for the dependence measure.

III) We obtain central limit theorem of empirical process under $\gamma-$weak dependence following from the tightness criterion given in Theorem 2.1 of Shao and Yu(1996).



2 Notation and Definitions

➢Wasserstein distance

Dfinition 1 For probability measures μ, ν on a separable metric space (U, d),the Wasserstein distance is defined as follows:

$$d_W(\mu,\nu) = \inf\{E[d(X,Y)]\}, \qquad (1)$$

where the inf is w.r.t. all random variables X, Y with distributions μ, ν. Kantorovich-Rubinstein theorem gives a dual representation of Wasserstein distance in terms of a Lipschitz-metric:

$$d_W(\mu,\nu) = \sup\left|\int f d(\mu-\nu)\right| \qquad (2)$$

i.e. sup over all f s.t. $|f(x)-f(y)| \leq d(x,y)$. d being the underlying metric on the space.

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➢γ− weak dependence coefficient

Definition 2 The sequence $\{X_n\}_{n\in N}$ is called γ−weak dependent, if there exists a sequence $\theta=(\theta_n)_{n\in N}$ decreasing to zero at infinity such that for any u-tuple $(i_1,\cdots,i_u)$ and any v-tuple $(j_1,\cdots,j_v)$ with $i_1\le\cdots\le i_u<i_u+r\le j_1\le\cdots\le j_v$, one has

$$d_W(\mu_{uv},\nu_{uv})=\sup_{\|f\|_L\le 1}\int f d(\mu_{uv}-\nu_{uv})\le(u+v)\theta_r. \quad (3)$$

where μ_{uv} denotes the joint distribution of $(X_{i_1},\cdots,X_{i_u},X_{j_1},\cdots,X_{j_v})$ and $\nu_{uv}=\nu_u\times\nu_v$, here,$\nu_u$ and ν_v are the marginal distributions of $(X_{i_1},\cdots,X_{i_u})$ and $(X_{j_1},\cdots,X_{j_v})$,respectively. The Lipschitz norm is defined as follows:

$$\| f \|_L=\sup_{(x_1,\cdots,x_u)\neq(y_1,\cdots,y_u)}\frac{|f(x_1,\cdots,x_u)-f(y_1,\cdots,y_u)|}{|x_1-y_1|+\cdots+|x_u-y_u|}.$$

3 Examples

Example 1 Consider real valued functional autoregressive model:

$$X_n = T(X_{n-1}) + \eta_n,$$

if $T : R \to R$,such that for some $0 \le c < 1, \forall u, u' \in R, |T(u) - T(u')| \le c|u - u'|$. $(\eta_n)_{n\in Z}$ is i.i.d. r.v.s., if $\{X_t\}$ initialized from stationary distribution X_0, and satisfy $E|X_0| < \infty$, then $(X_n)_{n\in Z}$ is $\gamma-$weak dependent.

In fact, let $(X'_{i_1}, \cdots, X'_{i_u})$ is independent and identical distributed with $(X_{i_1}, \cdots, X_{i_u})$, and $(X'_{i_1}, \cdots, X'_{i_u})$ is independent of $(\eta_n)_{n\in Z}$. Denote $X'_n = T(X'_{n-1}) + \eta_n, n \ge i_u + 1$, then

$$d_W(\mu_{uv}, \nu_{uv}) \le E|X'_{j_1} - X_{j_1}| + \cdots + E|X'_{j_v} - X_{j_v}| \le 2vc^r E|X_0|.$$

So we can take $\theta_n = 2c^r E|X_0|$.

Example 2 For non-strong mixing Markov chain

$$X_n = (X_{n-1} + \eta_n)/2,$$

where $(\eta_n)_{n\geq 0}$ is i.i.d., $P(\eta_0 = 0) = P(\eta_0 = 1) = 1/2$. If $E|X_0| < \infty$, then $(X_n)_{n\in N}$ is $\gamma-$weak dependent, here $\theta_r = 2E|X_0|\frac{1}{2^r}$。

Example 3 Consider random coefficient autoregressive model:

$$X_t = c(\eta_{t-1})X_{t-1} + g(\eta_{t-1}),$$

where $\{\eta_t\}_{t\geq 1}$ is i.i.d. r.v.s.and independent of X_0. If $\{X_t\}$ initialized from stationary distribution, $\rho = E|c(\eta_t)| < 1, E|X_0| < \infty$, then $\{X_t\}_{t\geq 1}$ is γ-weak dependent, where $\theta_n = 2E|X_0|\rho^n$.

Example 4 Let $(\varepsilon_i)_{i\in N}$ be a sequence independent random variables, F is a measurable fuction, $(X_i)_{i\in N}$ is a amrkov chain and defined as follows:

$$X_{n+1} = F(X_n, \varepsilon_{n+1}), \tag{4}$$

Suppose the initial distribution X_0 is independent of $(\varepsilon_i)_{i\in N}$, and F satisfy for some $a \geq 1, 0 < \alpha < 1$,

$$E|F(0,\varepsilon_1)|^a < \infty, E|F(x,\varepsilon_1) - F(y,\varepsilon_1)|^a \leq \alpha^a |x-y|^a. \quad (5)$$

Duflo show that under the condition (5) Markov chain $(X_i)_{i\in N}$ have stationary law with finite moment of order α. Suppose Markov chain $(X_i)_{i\in N}$ initialized from stationary law, if $E|X_0|^a < \infty$, then $\{X_t\}_{t\geq 1}$ is γ-weak dependent, here$\theta_n = 2(\alpha^a)^n E|X_0|^a$.

Example 5 A simple bilinear process with recurrence equation

$$X_t = aX_{t-1} + bX_{t-1}\eta_{t-1} + \eta_t.$$

If $\{X_t\}$ initialized from stationary law X_0, and satisfy $c = E|a + b\eta_0| < 1, E|X_0| < \infty$, then $\{X_t\}_{t\geq 1}$ is γ-waak dependent, here $\theta_n = 2c^n E|X_0|$.

4 An Empirical Process CLT

Let $\{Y_t\}_{t\in Z}$ be a real-valued stationary process. The empirical process of the sequence $\{Y_t\}_{t\in Z}$ at time n is defined as $\frac{1}{\sqrt{n}}E_n(x)$, where

$$E_n(x) = \sum_{k=1}^{n}(\mathbf{1}_{(Y_k\le x)} - P(Y_k \le x)).$$

We consider the following convergence result in the Skohorod space D([0, 1]) when the sample size n tends to infinity:

$$\frac{1}{\sqrt{n}}E_n(x) \xrightarrow{d} \bar{B}(x).$$

Here $\bar{B}(x)$ denotes a centered Gaussian process with covariance given by

$$E\bar{B}(x)\bar{B}(y) = \sum_{k=-\infty}^{\infty}(P(Y_0\le x, Y_k \le y) - P(Y_0\le x)P(Y_k\le y)).$$

Theorem Let$(Y_i)_{i \in Z}$ be a stationary $\gamma-$ weak dependent sequence, if there exists constant $C > 0, a > 5$ such that $\theta_r \le Cr^{-a}$. Then the following empirical functional convergence holds true in the Skohorod space of real-valued càdlàg functions on the unit interval, D([0, 1]):

$$\frac{1}{\sqrt{n}} E_n(x) \xrightarrow{d} \bar{B}(x).$$

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5 Several Key Lemmas

We use a quantile transform to obtain that the marginal distribution of this sequence is the uniform law on [0, 1].

Lemma 1 Assume that $(Y_i)_{i\in Z}$ is a stationary sequence and the density of Y_i are bounded by C_Y, For $s<t, s,t\in[0,1]$, denote $g_{t,s}(x)=\mathbf{1}\{x\le t\}-\mathbf{1}\{x\le s\}$. Let $\mathbf{i}=(i_1,\cdots,i_u)\in Z^u, \mathbf{j}=(j_1,\cdots,j_v)\in Z^v$, and $i_1\le\cdots\le i_u<i_u+r\le j_1\le\cdots\le j_v$. Let G and H be two bounded Lipschitz functions on R^u and R^v respectively. Denote $Y_{\mathbf{i}}=(Y_{i_1},\cdots,Y_{i_u})$. If $(Y_i)_{i\in Z}$ is $\gamma-$weak dependent, then

$$\begin{aligned}&|Cov(G(g_{t,s}(Y_{\mathbf{i}})),H(g_{t,s}(Y_{\mathbf{j}})))|\\ \le\ & 2\sqrt{2C_Y}(u+v)(\| H\|_\infty Lip(G)+\| G\|_\infty Lip(H))\theta_r^{\frac{1}{2}}.\end{aligned}$$

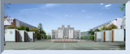
Lemma 2 If $(X_n)_{n\in N}$ is a sequence of centred r.v.s., then

$$ES_n^4 \leq 4!\{(n\sum_{r=0}^{n-1} C_{r,2}) + n\sum_{r=0}^{n-1}(r+1)^2 C_{r,4}\}.$$

Here,

$$C_{r,q} = \sup|Cov(X_{t_1}\cdots X_{t_m}, X_{t_{m+1}}\cdots X_{t_q})|,$$

where the supremum is taken over all $\{t_1, \cdots, t_q\}$ such that $1 \leq t_1 \leq \cdots \leq t_q$ and m, r satisfy $t_{m+1} - t_m = r$.



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Denote

$$S_n = n^{-\frac{1}{2}} \sum_{j\in\{0,\ldots,n\}} \alpha_j X_j.$$

Let $p(n)$ and $q(n)$ be sequences of integrers such that $p(n) = o(n), q(n) = o(p(n))$. Assume that $k = [n/(p+q)], r = n - k(p+q)$, for $i = 1, \cdots, k$we define the interval $P_i = \{(p+q)(i-1)+1, \cdots, (p+q)i - q\}$, if $r \neq 0$, $P_{k+1} = \{(p+q)k+1, \cdots, (p+q)k + r \vee p\}$.$Q$ the set of indices that are not in one of the P_i.Note that the cardinal of $Q$is less than kq. Denote $K = \{1, 2, \cdots, k+1\}$. For each block $P_i (1 \leq i \leq k+1)$ and Q, we define the partial sums:

$$u_i = \frac{1}{\sqrt{n}} \sum_{j\in P_i} \alpha_j X_j, \quad v = \frac{1}{\sqrt{n}} \sum_{j\in Q} \alpha_j X_j.$$

Lemma 3 Let $S_n = n^{-\frac{1}{2}} \sum\limits_{j\in\{0,\ldots,n\}} \alpha_j X_j$ be a sum of centred stationary r.v.s.. Set$\sigma_n^2 = VarS_n$. Suppose that:

(i)

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} Ev^2 = 0. \tag{6}$$

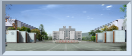
(ii)For any $t \in R$,

$$\sum_{j\in K} \left|Cov\left(g\left(\frac{t}{\sigma_n}\sum_{i\in K, i<j} u_i\right), h\left(\frac{t}{\sigma_n} u_j\right)\right)\right| \to 0. \tag{7}$$

where h and g be one of the trigonometric functions $x \to cosx, x \to sinx$.

(iii)For any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{i\in K} E|u_i|^2 I\{|u_i| \geq \varepsilon\sigma_n\} = 0 \tag{8}$$



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(iv)

$$\lim_{n\to\infty}\frac{1}{\sigma_n^2}\sum_{i=1}^{k}E|u_i|^2=1. \quad (9)$$

Then $\frac{S_n}{\sigma_n}$ converges in distribution to a Gaussian $N(0,1)-$distribution.

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Thank You !