Phase Transition on the Degree Sequence of a Mixed Random Graph Process

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- 1, Scale-Free Real-World Networks

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- 5, Our Model and Main results

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- 6, Comparing Argument

1, Scale-Free Real-World Networks

- For the real-world network of World Wide Web/Internet, experimental studies by Albert, Barabási & Jeong (1999) etc. demonstrated that the proportion of vertices of a given degree follows an approximate inverse power law, i.e.,

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- The degree distribution of real-world networks (Internet) is heavy-tailed.

- For the classical random graph model $G_{n,p}$ introduced by Erdös & Rényi (1959), the proportion of vertices of a given degree follows an approximate Poisson law, i.e.,

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- The degree distribution of the network of world airports Amaral *et al*. (2000) interpolates between Gaussian and exponential distributions.
- The degree distribution of the citation network in high energy physics Lehmann, Lautrup & Jackson (2003) interpolates between exponential and power law distributions.

An example: a model which exhibits more than one D.S.

- For a general model of collaboration networks in Zhou *et al.* (2005) indicate that:

  while a relevant parameter α increases from 0 to 1.5, four kinds of degree distributions appear as:

  1, exponential,

  2, arsy-varsy,

  3, semi-power law and

  4, power law

  in turn.

3, Models lead to Power Law D. S.

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- "hard copying" model of Ning, Wu & Cai (2008). etc.

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- Our goal:

  Answer the above problem in a mathematically rigorous manner.

The First Result (A simplified version!)

Model 1 [Wu, Dong, Liu and Cai (2008)]:

$\{G_t = (V_t, E_t), t \geq 1\}$, Write $e_t = |E_t|$, $v_t = |V_t|$.

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- Let $G_1 = \{x_1\}$
- At Time-Step $t \geq 2$, to define G_t from G_{t-1}, one of the two following substeps is executed.
- With probability $\alpha > 0$ we add a vertex x_t to G_{t-1}. We then add m random edges incident with x_t. When a edge is added, the random neighbour w of x_t is chosen in the manner of preferential attachment, namely,

$$\mathbb{P}(w = v) = \frac{d_v(t-1)}{2e_{t-1}},$$

where $d_v(t-1)$ denotes the degree of vertex v in G_{t-1}.

- With probability $1-\alpha \geq 0$ we delete $\min\{m, e_{t-1}\}$ randomly chosen edges from E_{t-1}.

Remark 1: This is the simplest case we have handled and we use it to state the result more clear.

Remark 2: In our setting, $\{e_t : t \geq 1\}$ is Markovian and

$$\mathbb{E}(e_t) \approx (2\alpha - 1)mt.$$

Now, Let $D_k(t)$ be the number of vertices with degree $k \geq 0$ in G_t and let $\overline{D}_k(t)$ be the expectation of $D_k(t)$. The main results for Model 1 follow as

- **Theorem**: Let $\alpha_c = \dfrac{2}{3}$, then it is a critical point for the degree sequence of the model satisfying:

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  1. if $\alpha > \alpha_c$, then there exists constant $C_1 = C_1(m, \alpha)$ such that,

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  2. if $\frac{4}{7} < \alpha < \alpha_c$, then there exists constant $C_2 = C_2(m, \alpha)$ such that

$$\lim_{t\to\infty} \frac{\overline{D}_k(t)}{t} = C_2 \gamma^k k^{-1+\beta} + O(\gamma^k k^{-2+\beta});$$

3 if $\alpha = \alpha_c$, then there exists constant $C_c = C_c(m, \alpha)$ such that,

$$\lim_{t \to \infty} \frac{\overline{D}_k(t)}{t} = C_c u_c(k).$$

- Where

$$u_c(k) = \int_0^1 t^{k-1} e^{-\frac{1}{1-t}} dt$$

and

$$\beta = \frac{4\alpha - 2}{3\alpha - 2}, \quad \gamma = \frac{\alpha}{2(1-\alpha)}.$$

- Remark 3: With help of computer calculation, $u_c(k)$ satisfies

$$\lim_{k\to\infty} \ln u_c(k)/(-k) = \lim_{k\to\infty} (-\ln k)/\ln u_c(k) = 0.$$

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- Model 1 exhibits critical phenomenon on its degree distribution!

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- To reconcile the ER theory of random graphs and various models of complex networks and develop a coherent or modern theory of random theory and complex networks.

The Model (Model 2)

Fix some constants $0 \leq \alpha \leq 1$ and $\mu,\ \zeta > 0$. Define random graph process $\{G_t^\alpha = (V_t, E_t) : t \geq 1\}$ as follows.

- *Time-Step 1.* Let G_1^α consists of vertices x_0, x_1 and the edge $\langle x_0, x_1 \rangle$.

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  2. with probability $1-\alpha$, we add random edges incident with x_t in the classical manner: for any $0 \le i \le t-1$, edge $\langle x_i, x_t \rangle$ is added independently with probability $(\zeta \wedge t)/t$.

Two Special Cases:

- *Case 1:* $\alpha = 0$: $\{G_t^0 : t \geq 1\}$ is an evolving version of the ER model and we call it classical process! Clearly, at each step, edges are added in an equal probability, this coincides with the essential feature of ER model $G_{n,p}$.

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- $\{G_t : t \geq 1\}$ is a good candidate which fits the two motivations of us.

Main Results

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  Theorem 1.1: For any $0 < \mu \leq 2$, there exists positive constants C_1 and C_2 such that

$$C_1 k^{-3} \leq \liminf_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq C_2 k^{-3}$$

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  Theorem 1.2: Assume that $0 < \mu \leq 2$. Then for any small enough $\nu > 0$, we have

$$\mathbb{E}(|C_t|) = (1 - e^{-\mu})t + O(t^{\frac{1}{2-\nu}}),$$

  where C_t be the giant component of G_t and $|G_t|$ be its size.

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  Theorem 1.3: For any $0 < \alpha < 1$, $0 < \mu \leq 2$ and $\zeta > 0$, there exists positive constants C_1^α and C_2^α such that

$$C_1^\alpha k^{-\beta} \leq \liminf_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq C_2^\alpha k^{-\beta}$$

for all $k \geq 1$, where $\beta = 1 + 2\left(1 + \frac{(1-\alpha)\zeta}{\alpha\mu}\right)$.

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  for all $k \geq 1$, where $\beta = 1 + 2\left(1 + \dfrac{(1-\alpha)\zeta}{\alpha\mu}\right)$.

  Remark 5: Note that at any Time-Step $t > \zeta$, the mean number of added new edges is $\xi := \alpha\mu + (1-\alpha)\zeta$ and $\dfrac{(1-\alpha)\zeta}{\alpha\mu}$ be the limit ratio of the number of the two kinds of edges in G_t^α.

Results for the classical process $\{G_t^0 : t \geq 1\}$:

- Theorem 1.4: For random graph process $\{G_t^0 : t \geq 1\}$, there exists positive constants C_1^0 and C_2^0 such that

$$C_1^0 \left(\frac{\zeta}{1+\zeta}\right)^k \leq \liminf_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq \limsup_{t\to\infty} \frac{\overline{D}_k(t)}{t} \leq C_2^0 \left(\frac{\zeta}{1+\zeta}\right)^k$$

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- Remark 7: Theorems 1.1, 1.3 and 1.4 exhibit a phase transition on the degree distributions of the mixed model $\{G_t^\alpha : t \geq 1\}$ while α varies from 0 to 1.

6, Comparing Argument (For model $\{G_t\}$)

By bounding e_t and Δ_t, the maximum degree of G_t properly, we can get the following recurrence for $\overline{D}_k(t)$:

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$$\begin{cases} \overline{D}_k(t+1) = \overline{D}_k(t) + \frac{k-1}{2}\frac{\overline{D}_{k-1}(t)}{t} - \frac{k}{2}\frac{\overline{D}_k(t)}{t} \\ \qquad\qquad +O(t^{-1/5}) + f_k(t), \;\; t+1 \geq k \geq 0, \;\; t \geq 1; \\ \overline{D}_0(1) = 0; \;\; \overline{D}_1(1) = 2; \;\; \overline{D}_k(t) = 0, \;\; k > t \geq 1; \\ \qquad\qquad \overline{D}_{-1}(t) = 0, \; t \geq 1. \end{cases}$$

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- While $f_k(t)$ is replaced by f_k, a real number, then the recurrence can be solved by a standard way. The main technique of this paper is to develop a comparing argument to solve the above recurrence.

- By studying the property of $f_k(t)$, the probability that exactly k edges are added at time t, we get its lower bound $\tilde{f}_k(t)$ and upper bounds $\widehat{f}_k(t)$. Then we prove that

$$\tilde{D}_k(t) \leq \overline{D}_k(t) \leq \widehat{D}_k(t), \quad \forall\, k \geq -1,\ t \geq 1,$$

where $\tilde{D}_k(t)$, $\widehat{D}_k(t)$ satisfying the above recurrence with $f_k(t)$ replaced by $\tilde{f}_k(t)$, $\widehat{f}_k(t)$ respectively.

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where $\tilde{D}_k(t)$, $\widehat{D}_k(t)$ satisfying the above recurrence with $f_k(t)$ replaced by $\tilde{f}_k(t)$, $\widehat{f}_k(t)$ respectively.

- $\tilde{f}_k(t)$ and $\widehat{f}_k(t)$ have the following form:

$$\tilde{f}_k(t) = \begin{cases} 0, & k \geq 1,\ t \geq 1, \\ \tilde{f}_k, & k = 0,\ t \geq 1; \end{cases} \quad \widehat{f}_k(t) = \begin{cases} \widehat{f}_k, & t \geq k, \\ 0, & 1 \leq t < k. \end{cases}$$

- And $\tilde{f}_k$, $\widehat{f}_k$ have the following form:

$$\tilde{f}_k = \begin{cases} 0, & k \geq 1, \\ \rho, & k = 0; \end{cases} \quad \text{and} \quad \widehat{f}_k = \begin{cases} Ck^{-4}, & k \geq 1, \\ e^{-\mu}, & k = 0, \end{cases}$$

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  where $\rho > 0$ be a constant.

- Using a standard argument, we can prove the following:

$$\lim_{t \longrightarrow \infty} \frac{\tilde{D}_k(t)}{t} = \tilde{d}_k, \qquad \lim_{t \longrightarrow \infty} \frac{\widehat{D}_k(t)}{t} = \widehat{d}_k.$$

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$$\begin{cases} \tilde{d}_k = \dfrac{k-1}{2}\tilde{d}_{k-1} - \dfrac{k}{2}\tilde{d}_k + \tilde{f}_k, & k \geq 0, \\ \tilde{d}_{-1} = 0; \end{cases}$$

- Where $\tilde{d}_k$, $\widehat{d}_k$ be the solutions of the following recurrences in k respectively:

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$$\begin{cases} \widehat{d}_k = \frac{k-1}{2}\widehat{d}_{k-1} - \frac{k}{2}\widehat{d}_k + \widehat{f}_k, & k \geq 0, \\ \widehat{d}_{-1} = 0. \end{cases}$$

- The recurrence in k with the form

$$\begin{cases} d_k = \frac{k-1}{2} d_{k-1} - \frac{k}{2} d_k + \phi_k, \quad k \geq 0, \\ d_{-1} = 0; \end{cases}$$

can be directly solved as: $d_{-1} = 0$, $d_0 = \phi_0$, $d_1 = \frac{2}{3}\phi_1$ and

$$d_k = \sum_{j=1}^{k} \frac{2j(j+1)}{k(k+1)(k+2)} \phi_j = \frac{1}{k(k+1)(k+2)} \sum_{j=1}^{k} 2j(j+1)\phi_j,$$

for all $k \geq 2$. Applied to $\{\tilde{f}_k\}$ and $\{\widehat{f}_k\}$, the summation in the right hand side of the above equation converges as $k \to \infty$, thus, $\tilde{d}_k$ and $\widehat{d}_k$ decay as k^{-3}.

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3. $\tilde{d}_k$ and $\widehat{d}_k$ decay as k^{-3}.

- We then finish the comparing argument and get Theorem 1.1. Namely, for some constants C_1 and C_2,

$$C_1 k^{-3} \le \liminf_{t \to \infty} \frac{\overline{D}_k(t)}{t} \le \limsup_{t \to \infty} \frac{\overline{D}_k(t)}{t} \le C_2 k^{-3}$$

for all $k \ge 1$.

- Thank You Very Much!