Differential Harnack inequality and Perelman's entropy formula on complete Riemannian manifolds

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Conjecture (H. Poincaré 1904)

Every compact and simply connected 3*-dimensional (smooth) manifold is homeomorphic (diffeomorphic) to* S^3.

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In August 2006, Perelman was awarded the Fields medal at ICM 2006 Madrid. He refused to receive it.

Hamilton's Ricci flow (RF)

Let M be a compact manifold with a Riemannian metric g. Let $g(t)$ be the solution of the Ricci flow equation given by

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}.$$

More precisely, for all $i, j = 1, \ldots, n$,

$$\frac{\partial g_{ij}(t)}{\partial t} = -2\mathrm{R_{ij}}(g(t)),$$

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More precisely, for all $i, j = 1, \ldots, n$,

$$\frac{\partial g_{ij}(t)}{\partial t} = -2\mathrm{R_{ij}}(g(t)),$$

which is a nonlinear 2*nd* order weakly parabolic equation of systems,

where $Ric = (R_{ij})$ =the Ricci tensor of g. The scalar curvature of $g(t)$ satisfies the nonlinear reaction-diffusion heat equation

$$\frac{\partial R}{\partial t} = \Delta R + |Ric|^2.$$

Hamilton's theorems

Theorem (Hamilton 1982)

Given a compact Riemannian manifold (M, g_0)*, there exists a* $T > 0$ *such that the Ricci flow equation*

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}, \quad t \geq 0$$

has a unique solution $g(t, x)$ *in* $[0, T) \times M$ *such that*

$$g(0) = g_0.$$

Hamilton's theorems

Theorem (Hamilton 1982)

Let M be a 3-dimensional compact manifold, g_0 a Riemannian metric on M with positive Ricci curvature. Then the normalized Ricci flow equation

$$\frac{\partial}{\partial t} g(t) = \frac{2r}{n} g(t) - 2\mathrm{Ric}_{g(t)},$$

where

$$r = \frac{\int_M R}{V(M)},$$

has a global solution $g(t)$ on $[0, \infty) \times M$ such that

$$g(0) = g_0.$$

Moreover, $g(t)$ converges to a Riemannian metric of constant positive Ricci (and hence sectional) curvature.

Perelman's modified Ricci flow (MRF)

In 2002, G. Perelman modified Hamilton's Ricci flow equation.
Let $\mathcal{M} = \{g : \text{Riemannian metrics on } M\}$. Define

$$\mathcal{F} : \mathcal{M} \times C^{\infty}(M) \to \mathbb{R}$$

Perelman's modified Ricci flow (MRF)

In 2002, G. Perelman modified Hamilton's Ricci flow equation.
Let $\mathcal{M} = \{g : \text{Riemannian metrics on } M\}$. Define

$$\mathcal{F} : \mathcal{M} \times C^\infty(M) \to \mathbb{R}$$

$$\mathcal{F}(g, f) := \int_M (R + |\nabla f|^2) e^{-f} dv,$$

$$R = \mathrm{Tr} Ric = \text{the scalar curvature of } g.$$

Theorem (Perelman 2002 Arxiv)

The gradient flow of $\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, with condition

$$dm = e^{-f}\sqrt{detg}dx \text{ being fixed,}$$

is given by the modified Ricci flow (MRF)

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2(Ric_g + \nabla^2 f), \\ \frac{\partial}{\partial t} f &= -\Delta f - R. \end{aligned}$$

Perelman's modified Ricci flow

Theorem (Perelman 2002 Arxiv)

Let $(g(t), f(t))$ be the solution of the Ricci flow (obtained via a time-dependent change of diffeomorphism on (MRF))

$$\begin{aligned} \partial_t g &= -2Ric_g, \\ \partial_t f &= -\Delta f + |\nabla f|^2 - R. \end{aligned}$$

Then

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |Ric + \nabla^2 f|^2 e^{-f} dv.$$

In particular, $\mathcal{F}(g(t), f(t))$ is nondecreasing in time and the monotonicity is strict except that

$$Ric + \nabla^2 f = 0 \quad \text{(steady Ricci soliton)}.$$

Perelman's modified Ricci flow

(M, g) is called a Ricci soliton if there exist a function $f \in C^\infty(M)$ and some $\lambda \in \mathbb{R}$ such that

$$Ric + \nabla^2 f = \lambda g,$$

$$\lambda > 0, \quad \text{shrinkingRiccisoliton}$$

$$\lambda = 0, \quad \text{steady Ricci soliton}$$

$$\lambda < 0, \quad \text{expandingRiccisoliton}.$$

Theorem (Hamilton 95, Ivey 93)

Every compact Riemannian Ricci steady or expanding soliton must be Einstein.

Perelman's $\mathcal{W}$-entropy functional

To study shrinking soliton, Perelman introduced the following important entropy functional

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

Theorem (Perelman 2002 Arxiv)

Let $g(t), f(t), \tau(t)$ *be the solution of*

$$
\begin{aligned}
\partial_t g &= -2Ric, \\
\partial_t f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\
\partial_t \tau &= -1.
\end{aligned}
$$

Then

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv = constant,$$

and

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M \left| Ric + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

In particular, $\mathcal{W}(g, f, \tau)$ is nondecreasing in time and the monotonicity is strict unless that (M, g) is a shrinking Ricci soliton

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

Perelman's $\mathcal{W}$-entropy functional

What is the hidden insight when Perelman introduced the W-entropy functional? Is there some relationship between Perelman's entropy and Boltzmann's entropy?

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

What is the role of the Gaussian heat kernel in Perelman's W-entropy functional?

What is the role of the dimension $n = \mathrm{dim} M$ in Perelman's W-entropy functional?

Bakry-Emery Ricci tensor 1984

Let M be a Riemannian manifold, $\phi \in C^\infty(M)$. Let

$$L = \Delta - \nabla\phi \cdot \nabla,$$

and

$$d\mu = e^{-f} dv$$

Then, $\forall f, g \in C_0^\infty(M)$, it holds

$$\int_M \langle \nabla f, \nabla g \rangle d\mu = \int_M (-Lf) g d\mu = \int_M f(-Lg) d\mu.$$

In 1984, Bakry and Emery introduced the notion of Ricci tensor associated with L on (M, g) is defined by

$$Ric(L) = Ric + \nabla^2\phi.$$

Bakry-Emery Ricci tensor associated to OU operator

On $\mathbb{R}^n$ with standard Gaussian measure

$$d\gamma_n(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}dx,$$

we have

$$L = \Delta - x \cdot \nabla,$$

and

$$Ric(L) = Ric + \nabla^2 f = \frac{g}{2t}.$$

Entropy functional for linear heat equation

Recall Perelman's W-entropy functional for Ricci flow:

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

Let (M, g) be a compact Riemannian manifold. When the Riemannian metric on M does not change, L. Ni (2004) studied the monotonicity of the following W-entropy functional

$$\mathcal{W}(f, \tau) = \int_M \left[\tau(|\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

Entropy formula for heat equation

Let u be a positive solution of

$$(\partial_t - \Delta)u = 0.$$

Denote

$$u = \frac{e^{-f}}{(4\pi t)^{n/2}}.$$

Inspired by Perelman's work, L. Ni (2004) first observed that the W-entropy functional can be understood in the following way:

Theorem (Ni 2004)

Let

$$H_n(u,t) = \int_M u \log u dv - \left(\frac{n}{2}\log(4\pi t) + \frac{n}{2}\right),$$

$$\mathcal{W}(u,t) = \int_M \left(t|\nabla f|^2 + f - n\right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv.$$

Then

$$\frac{d}{dt} H_n(u,t) = -\int_M \left(\Delta \log u + \frac{n}{2t}\right) u dv,$$

$$\mathcal{W}(u,t) = \frac{d}{dt}(tH(u,t)).$$

Theorem (P. Li-S.T. Yau 1986)

Let $u > 0$ *be a positive solution of the heat equation*

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0.$$

Suppose that

$$Ric \geq 0.$$

Then the Li-Yau Harnack differential inequality holds

$$\Delta \log u + \frac{n}{2t} \geq 0.$$

Equivalently, we have

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}.$$

In the case $M = \mathbb{R}^n$, *we have*

$$\Delta \log u + \frac{n}{2t} = 0.$$

Theorem (Ni 2004)

Let $u(x,t)$ be a positive solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0.$$

Let f, τ be defined by

$$u(t,x) = \frac{e^{-f}}{(4\pi\tau)^{n/2}}, \quad \frac{d\tau}{dt} = 1.$$

Then

$$\frac{d}{dt}\mathcal{W}(f,\tau) = -2\int_M \tau \left(\left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + Ric(\nabla f, \nabla f) \right) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

In particular, if M has non-negative Ricci curvature, i.e., $Ric \geq 0$, then $\mathcal{W}(f,\tau)$ is decreasing along the heat equation.

Entropy functional for diffusion operator

Let (M, g) be a compact Riemannian manifold, $\phi \in C^2(M)$. Let

$$L = \Delta - \nabla\phi \cdot \nabla, \quad d\mu = e^{-\phi} dv.$$

Let

$$u = \frac{e^{-f}}{(4\pi t)^{m/2}}$$

be a positive solution of

$$(\partial_t - L)u = 0.$$

Inspired by the works of Perelman and Ni, we have the following

Entropy functional for diffusion operators

Theorem (X.-D. Li 2006)

Let

$$\begin{aligned} H_m(u,t) &= \int_M u \log u d\mu - \left(\frac{m}{2}\log(4\pi t) + \frac{m}{2}\right), \\ \mathcal{W}(u,t) &= \int_M \left(t|\nabla f|^2 + f - m\right) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu. \end{aligned}$$

Then

$$\frac{d}{dt}H_m(u,t) = -\int_M \left(L\log u + \frac{m}{2t}\right) u d\mu,$$

$$\mathcal{W}(u,t) = \frac{d}{dt}(tH(u,t)).$$

Theorem (Li-Yau 86, ..., X.-D. Li 05, Bakry-Ledoux 06)

Let u be a positive solution of the heat equation

$$\left(\frac{\partial}{\partial t} - L\right) u = 0.$$

Suppose that

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq 0.$$

Then the Li-Yau Harnack differential inequality holds

$$L \log u + \frac{m}{2t} \geq 0.$$

Equivalently, we have

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}.$$

In the case $M = \mathbb{R}^n$, $\phi = 0$, we have

$$L \log u + \frac{m}{2t} = 0.$$

Theorem (X.-D. Li 2006)

Let u be a positive solution of the heat equation

$$(\partial_t - L)\, u = 0.$$

Let f be defined by

$$u(t,x) = \frac{e^{-f}}{(4\pi\tau)^{m/2}}.$$

Then

$$\begin{aligned}\frac{d\mathcal{W}(u,t)}{dt} &= -2\int_M \tau\left(\left|\nabla^2 f - \frac{g}{2\tau}\right|^2 ud\mu + Ric_{m,n}(L)(\nabla f, \nabla f)\right) ud\mu \\ &\quad -\frac{2}{m-n}\int_M \tau\left(\nabla\phi\cdot\nabla f + \frac{m-n}{2\tau}\right)^2 ud\mu.\end{aligned}$$

Theorem (X.-D. Li 2006)

Suppose that there exists a constant $m \geq n$ *such that*

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geq 0.$$

Then $\mathcal{W}(u,t)$ *is monotone decreasing along the heat equation* $(\partial_t - L)u = 0$*, i.e.,*

$$\frac{d\mathcal{W}(u,t)}{dt} \leq 0.$$

We introduce the optimal constant in the Log-Sobolev inequality

$$\begin{aligned}\mu(\tau) &= \inf_{\int_M u d\mu = 1} \mathcal{W}(u,\tau) \\ &= \inf \left\{ \int_M \left[4\tau|\nabla u|^2 - u^2 \log u^2 - mu^2 \right] \frac{d\mu}{(4\pi\tau)^{m/2}} \right\}\end{aligned}$$

where inf is taken among all the u such that

$$\int_M \frac{u^2}{(4\pi\tau)^{m/2}} d\mu = 1.$$

Corollary (X.-D. Li 2006)

Suppose that $Ric_{m,n}(L) \geq 0$. Then $\tau \mapsto \mu(\tau)$ is decreasing along the heat diffusion $(\partial_\tau - L)u = 0$.

Li-Yau-Hamilton-Perelman Harnack inequality

Let

$$\mathcal{W}(g,f,\tau)=\int_M\left[\tau(R+|\nabla f|^2)+f-n\right]\frac{e^{-f}}{(4\pi\tau)^{n/2}}dv,$$

and

$$\nu=[\tau(2\Delta f-|\nabla f|^2+R)+f-n]u.$$

Then

$$W(g,f,\tau)=\int_M \nu dv,$$

$$\frac{d}{dt}W(g,f,\tau)=-\int_M \Box^*\nu dv,$$

where

$$\Box^*=-\frac{\partial}{\partial t}-\Delta+R.$$

In 2002, Perelman proved a Li-Yau-Hamilton Harnack inequality for the fundamental solution of the conjugate backward heat equation of the Ricci flow.

Li-Yau-Hamilton-Perelman Harnack inequality

Theorem (Perelman 2002)

Let $g(t)$ be the solution to the Ricci flow on $M \times (0, T)$, i.e.,

$$\partial_t g = -2Ric_g.$$

Let

$$H = \frac{e^{-f}}{(4\pi t)^{m/2}}$$

be the fundamental solution to the conjugate backward heat equation

$$\partial_t u = -\Delta u - Ru.$$

Then

$$\nu_H = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]H \leq 0.$$

In 2004/2006, Ni proved the Li-Yau-Hamilton-Perelman Harnack for the heat equation $(\partial_t - \Delta)u = 0$ with $Ric \geq 0$:

$$\nu_H = [\tau(2\Delta f - |\nabla f|^2) + f - n]H \leq 0,$$

where

$$H = \frac{e^{-f}}{(4\pi t)^{m/2}}$$

is the fundamental solution to the heat equation

$$\partial_t u = \Delta u.$$

Conjecture

Let M be a complete Riemannian manifold, $\phi \in C_b(M) \cap C^2(M)$ be bounded C^2-function. Let

$$H = \frac{e^{-f}}{(4\pi t)^{m/2}}$$

be the fundamental solution to the heat equation

$$\partial_t u = Lu.$$

Suppose that

$$Ric + \nabla^2\phi \geq 0.$$

Then, there exists $t_0 > 0$ such that, for all $t > t_0$, the Li-Yau-Hamilton-Perelman Harnack inequality holds:

$$\nu_H = [\tau(2Lf - |\nabla f|^2) + f - m]H \leq 0.$$

Difficulty

Following Perelman's argument, let $h(t), t \in [0, T]$ be the positive solution of the heat equation

$$\partial_t h = -Lh, \quad h(T) = h.$$

Then

$$\frac{d}{dt}\int_M h(t)v_H(t)d\mu \leq 0.$$

To prove the above conjecture, it is enough to verify that

$$\liminf_{t \to 0} W_h(t) \leq 0$$

for all positive continuous function $h > 0$ on M, where

$$W_h(\tau) = \int_M h(\tau)\nu_H(\tau)dv.$$

$$W_h(\tau) = \int_M \left[\tau\left(\frac{|\nabla H|^2}{H^2} - 2\frac{LH}{H}\right) - \left(\log H + \frac{m}{2}\log(4\pi\tau) + m\right)\right] Hhd\mu$$

By Li-Yau Harnack inequality, if $Ric_{m,n}(L) \geq 0$, then

$$\frac{|\nabla H|^2}{H^2} - \frac{LH}{H} \leq \frac{m}{2\tau}.$$

Thus

$$W_h(\tau) \leq -\tau \int LHhd\mu - \int \left(\log H + \frac{m}{2}\log(4\pi\tau) + \frac{m}{2}\right) Hhd\mu.$$

The difficulty is to obtain the Gaussian lower bound estimate of the heat kernel for symmetric diffusion operators on complete Riemannian manifolds with weighted volume measure.

Remark

Let $M = \mathbb{R}^n$, $\phi = 0$, $m > n$. Then $Ric + \nabla^2\phi = 0$. In this case, we can verify that

$$\begin{aligned} W_h(\tau) &= (n-m)\int_{\mathbb{R}^n} H(x,y,\tau)h(y,t)dy \\ &\quad -\frac{m-n}{2}\int_{\mathbb{R}^n} \log(4\pi\tau)H(x,y,\tau)h(y,\tau)dy. \end{aligned}$$

We can verify that the above conjecture is true with a certain $t_0 > 0$ but it can not be true for $t_0 = 0$. Indeed, we have

$$\liminf_{t\to 0} W_h(t) = +\infty.$$

Question

What happens when $Ric_{m,n}(L) \geq 0$ but ϕ is unbounded?

Harnack inequality to conjugate heat equation

Let $g(t), t \in [0, T)$ be the solution to the Ricci flow, u be a positive solution to the conjugate heat equation

$$\begin{aligned} \partial_t g &= -2Ric_g, \\ \partial_t u &= -\Delta u - Ru. \end{aligned}$$

Let $\tau = T - t$. Define

$$H_n(u,\tau) = \int_M u \log u dv - \left(\frac{n}{2}\log(4\pi\tau) + \frac{n}{2}\right).$$

By Perelman (2002) we have

$$\frac{dH_n}{d\tau} = -\int_M \left(\Delta \log u - R + \frac{n}{2\tau}\right) u dv.$$

Moreover, following Ni (2004) and Topping (2006),

$$W(u,\tau) = \frac{d}{d\tau}(\tau H_n(u,\tau)).$$

Differential Harnack inequality for Ricci flow

Recently, several authors (Kuang-Zhang 07, Cao 08, etc) have studied the Li-Yau-Hamilton differential Harnack inequality for the conjugate heat equations. The following differential Harnack inequality has been proved very recently.

Theorem (X.-D. Li 2008)

Let $g(t)$ be the solution to the Ricci flow on an n-dimensional compact manifold M and on $[0, T)$. Let u be a positive solution to the forward heat equation

$$\partial_t u = \Delta u - Ru.$$

Suppose that $(M, g(t))$ has non-negative Ricci curvature. Then, there exists a constant $C > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - R \leq \frac{n(1+\varepsilon)}{2\tau} + \sqrt{n(1+\varepsilon^{-1})C}.$$

Recall that Perelman (2002) has proved

$$\frac{dH_n}{d\tau} = -\int_M \left(\Delta \log u - R + \frac{n}{2\tau}\right) u dv.$$

To end this talk, I would like to raise the following

Question

Under which condition, can we prove the following differential Harnack inequality

$$\Delta \log u - R + \frac{n}{2\tau} \geq 0 \quad ?$$

Thank you !

Thank you !