Spectral Analysis of Brownian Motion with Jump Boundary

Wenbo V. Li
University of Delaware
E-mail: wli@math.udel.edu
Wuhu, July 2008

Consider a family of probability measures $\{\nu_y : y \in \partial D\}$ on a bounded open domain $D \subset \mathbb{R}^d$ with smooth boundary. For any starting point $x \in D$, we run a a standard d-dimensional Brownian motion $B(t) \in \mathbb{R}^d$ until it first exits D at time τ, at which time it jumps to a point in the domain D according to the measure $\nu_{B(\tau)}$ at the exit time, and starts the Brownian motion afresh. The same evolution is repeated independently each time the process reaches the boundary. The resulting diffusion process is called Brownian motion with jump boundary (BMJ). The spectral gap of non-self-adjoint generator of BMJ, which describes the exponential rate of convergence to the invariant measure, is studied. The main analytic tool is Fourier transforms with only real zeros. This is a joint work with Yuk Leung and Rakesh to appear in Proc. AMS.

Motivations

There are various motivating applications for the study of the BMJ process (also called rebirth process in Grigorescu and Kang (2002)).

•A variant of the Fleming-Viot branching process introduced by Burdzy, Holyst and March (2000).

•Connected to the study of the behavior of the double knock-out barrier options in derivative markets in mathematical finance, Grigorescu and Kang (2002, 03, 03).

•Related to the study of Brownian flow on a finite interval with jump boundary conditions, Kosygina (2006).

•Ergodicity of non-reversible Markov processes, Ben-Ari and Pinsky (2007, 2007+).

•MCMC and Fourier transforms with only real zeros, Leung, Li and Rakesh (2008).

•Neuron firing models in mathematical biology.

Known Approaches on Spectral Analysis of BMJ

• The jump measures $\nu_y = \delta_p$ for all $y \in \partial[0,1] = \{0,1\}$ in $\mathbb{R}$, i.e. the jumps are deterministic and concentrated on a single point $p \in (0,1)$, the ergodicity of BMJ was studied by Grigorescu and Kang (2002, 06). The main tools used are Laplace transform methods and the theory of analytic semigroups.

• The jump measures $\mu_y = \mu$ for all $y \in \partial D$, i.e. the jumps follows the same measure ν on D, the ergodicity of BMJ (as an interesting special case) was systematically studied in Ben-Ari and Pinsky (2007). They used a powerful functional analytic approach.

• The jump measures ν_y various for $y \in \partial D$, see Ben-Ari and Pinsky (2007+) and Leung, Li and Rakesh (2008).

The key point of these papers is to give a formula for the invariant probability measure and to describe the exponential rate at which the distribution of the process converges to this invariant measure in terms of the spectral gap of the generator of BMJ. Note that BMJ is *never* reversible and hence powerful methods developed recently for ergodic convergence rates of reversible Markov processes cannot be applied directly.

The Same Measure Case: $\nu_y = \nu$ for All $y \in \partial D$

If $p(t,x,\cdot)$ represents the transition probability measure for the BMJ with the same jump measure ν, then it was shown in [BP07] that $p(t,x,\cdot)$ approaches a unique stationary invariant measure μ on D with the density

$$\mu(dy) = \frac{\int_D G^D(x,y)d\nu(x)dy}{\int_D \int_D G^D(x,y)d\nu(x)dy} = \frac{G^D(\nu,y)dy}{\int_D G^D(\nu,y)dy}.$$

Here, for every $x \in D$, the Green's function $G^D(x,y)$ is the solution of

$$\begin{aligned} \frac{1}{2}\Delta_x G^D(x,y) &= -\delta(x-y) \qquad && x \in D \\ G^D(x,y) &= 0 \qquad && x \in \partial D. \end{aligned}$$

Equivalently, $G^D(x,y) = \int_0^\infty p^D(t,x,y)dt$ is the 0-potential of the transition sub-probability function $p^D(t,x,y)$ of the absorbed Brownian motion on D.

Earlier Results on Spectral Analysis of BMJ

Consider the eigenvalue problem (with a nonlocal boundary condition)

$$\frac{1}{2}\Delta u = \lambda u \quad \text{in } D$$
$$u|_{\partial D} = \int_D u \, d\nu.$$

Clearly 0 is an eigenvalue for this problem; define the spectral gap

$$\lambda_1(\nu) = \sup\{\Re\lambda : 0 \neq \lambda \text{ is an eigenvalue of equations above }\}.$$

Then [BP07] also characterized the rate of decay of $p(t, x, \cdot)$ to μ as

$$\lim_{t\to\infty} \frac{1}{t} \sup_{f\in L^\infty(D), \|f\|_\infty \leq 1} \log \|\mathbb{E}_x f(X(t)) - \int_D f d\mu\|_\infty = \lambda_1(\nu) < 0$$

Two Jump Measures

We now focus our attention on the BMJ when $D = (a, b)$ with the jump measures ν_a at a and ν_b at b. An important role is played by the eigenvalues of the nonlocal eigenvalue problem

$$\frac{1}{2}u'' = \lambda u \quad \text{on } (a, b)$$

$$u(a) = \int_a^b u(x)\, d\nu_a(x), \qquad u(b) = \int_a^b u(x)\, d\nu_b(x).$$

•Note that, for $D = (a, b)$, we have the well known formula

$$G^D(x, y) = \frac{2}{b-a}(b - \max(x, y))\,(\min(x, y) - a), \qquad x, y \in (a, b).$$

Thm: (LLR08). Let $X(t)$ be the BMJ process on (a,b) associated to the probability measures ν_a, ν_b and let m_a, m_b be the means of ν_a, ν_b. Then $X(t)$ has a unique invariant measure μ given by

$$d\mu(y) = \frac{(b-m_b)G^D(\nu_a,y) + (m_a-a)G^D(\nu_b,y)}{(b-m_b)\int_a^b G^D(\nu_a,y)dy + (m_a-a)\int_a^b G^D(\nu_b,y)dy}$$

and the rate of convergence to μ is characterized by

$$\lim_{t\to\infty}\frac{1}{t}\sup_{f\in L^\infty(a,b),\|f\|_\infty\le 1}\log\|\mathbb{E}_x f(X(t)) - \int_a^b f d\mu\|_\infty = \lambda_1(\nu_a,\nu_b) < 0$$

where (the spectral gap)

$$\lambda_1(\nu_a,\nu_b) = \sup\{\Re\lambda : 0 \ne \lambda \text{ is an eigenvalue of equations above }\}.$$

•The invariant measure is a “mixed” Green’s function (normalized). via associated “mixed” harmonic measure.

Earlier Results on Spectral Gap of BMJ

•If the jump measure ν is such that the nonzero eigenvalue with the largest real part is real, then

$$\lambda_1(\nu) < \lambda_0^D < 0$$

where λ_0^D is the principal eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary condition.

•If the jump measure $\nu = m_d$ =Lebesgue measure on the cube domain $D = (0,1)^d$, then

$$\lambda_1(m_d) = \lambda_1^D = -\frac{(d+3)\pi^2}{2} \quad \text{for} \quad d \leq 10;$$

and

$$\lambda_0^D = -\frac{d\pi^2}{2} > \lambda_1(m_d) > -\frac{(d+3)\pi^2}{2} = \lambda_1^D \quad \text{for} \quad d \geq 11.$$

where $\lambda_1^D < \lambda_0^D$ is the second eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary condition.

•There is a deterministic jump measure $\nu = \delta_a$ on the cube domain $D = (0,1)^2$ with $a \in (0,1)^d$ such that $\lambda_1(\delta_a) < \lambda_1^D = -5\pi^2/2$.

•If the single jump measure $\nu = \delta_p$, $0 < p < 1$, is a point measure in one-dimensional interval domina $D = (0, 1)$, then $\lambda_1(\delta_p) = -2\pi^2 = \lambda_1^D$.

•For all single jump measure ν in one-dimension interval domain $(0, 1)$, $\lambda_1(\nu) \geq \lambda_1^D = -2\pi^2$.

•If the nonzero eigenvalue with the largest real part is real for the jump measure ν in one-dimension interval domain $D = (0, 1)$, then $\lambda_1(\nu) = -2\pi^2 = \lambda_1^D$.

Ex-Conjecture: For one-dimension interval domain $D = (0, 1)$

$$\lambda_1(\nu) = -2\pi^2 = \lambda_1^D$$

for **all** single jump measure ν, see Ben-Ari and Pinsky (2007).

Main Results in LLR (2008)

Thm 1. For two jump measures ν_0 and ν_1 in one-dimension interval domain $D=(0,1)$, all eigenvalues associated with

$$\frac{1}{2}\Delta u=\lambda u \quad \text{in} D, \quad u(0)=\int_D u d\nu_0, \quad u(1)=\int_D u d\nu_1$$

are real and non-positive. As a consequence

$$\sup_{\nu_0,\nu_1}\lambda_1(\nu_0,\nu_1)=\lambda_0^D=-\frac{\pi^2}{2}$$

and if $\nu_0=\nu_1=\nu$ then

$$\lambda_1(\nu,\nu)=\lambda_1^D=-2\pi^2.$$

Here λ_0^D, λ_1^D are the largest and the second largest Dirichlet eigenvalues for $\Delta/2$ on $(0,1)$.

•**The** $2/3$ **Conj:**

$$\inf_{\nu_0,\nu_1}\lambda_1(\nu_0,\nu_1)=\lambda_2^D=-\frac{9\pi^2}{2}$$

where λ_2^D is the third largest Dirichlet eigenvalue; The equality can be attained at $\nu_0=\delta_{2/3}$ and $\nu_1=\delta_{1/3}$.

•**Convexity Conj:** For all $h \in [0,1]$

$$h\lambda_1(\nu_0,\nu_1) + (1-h)\lambda_1(\nu_0',\nu_1') \le \lambda_1(h\nu_0 + (1-h)\nu_0',\, h\nu_1 + (1-h)\nu_1').$$

Thm 2. For single jump measure ν on the open unit ball $B = \{|x| < 1\}$ in $\mathbb{R}^d$ ($d > 1$ odd) such that $r^{-d}\mu(\{x \in R^d : |x| < r\})$ is an increasing function of r on $[0,1)$, the eigenvalue problem

$$\frac{1}{2}\Delta u = \lambda u \quad \text{in } B, \quad u|_{\partial B} = \int_B u(x) d\nu$$

has only real eigenvalues.

•For single point measure δ_p with $|p| = 1/4$ on B in $\mathbb{R}^3$, there are complex eigenvalues.

Conj. of LLR (2008): The principle eigenvalue for BMJ is always real.

Why $\lambda_1(\nu) = -2\pi^2$ if it is real?

Every solution of the equation $\frac{1}{2}u'' = \lambda u$ is of the form

$$u(x) = A\cos(zx) + B\sin(zx),$$

where $\lambda = -z^2/2$ for some $z \in \mathcal{C}$. Note that here λ, A, B and z are, in general, complex numbers. The boundary condition $u(0) = u(1) = \int_0^1 u(x)d\nu(x)$ can be rewritten as

$$A\int_0^1 (1-\cos(zx))d\nu(x) - B\int_0^1 \sin(zx)d\nu(x) = 0$$
$$A(1-\cos z) - B\sin z = 0$$

Set the determinant to be zero, we obtain

$$\int_0^1 \sin\frac{z(1-x)}{2}\cdot\sin\frac{z}{2}\cdot\sin\frac{zx}{2}d\nu(x) = 0$$

Assume z is real. Since $\sin\frac{z(1-x)}{2}\cdot\sin\frac{z}{2}\cdot\sin\frac{zx}{2} > 0$, for $z \in (0, 2\pi)$ and $x \in (0,1)$, while the reverse inequality holds for $z \in (-2\pi, 0)$ and $x \in (0,1)$, the integral equation has no solution for $z \in (-2\pi, 2\pi) - \{0\}$. On the other hand, $\sin\frac{z(1-x)}{2}\cdot\sin\frac{z}{2}\cdot\sin\frac{zx}{2} = 0$ for $z = \pm 2\pi$. Thus, $\lambda = z^2/2 = 2\pi^2$ is the smallest positive nonzero solution to the eigenvalue problem. The case $\nu = \delta_p$ is simple.

Fourier Transforms with Only Real Zeros

Pólya (1918) showed various finite Fourier transforms which are entire functions vanish only on the real axis. This seminal paper has led to a host of interesting problems in combinatorics and in probability, such as the decomposition of the hypergeometric r.v.

In particular, Pólya showed that both real and imaginary parts of the finite Fourier transform of a non-negative non-decreasing function $f(x)$ on $[0,1]$ defined by $P(t)+iQ(t) := \int_0^1 e^{ixt} f(x)dx$ have only real zeros. Later, Szegö (1936) refined Pólya's method and showed that the zeros of an arbitrary real combination of P and Q are all real and they distribute regularly on the real line.

In addition, for distribution function $F(t)$ with $F(0) = 0$, $F(1) = 1$, the function $\int_0^1 \cos(zt)F(t)dt$ has only real roots iff $\sum_{k=0}^{n} \frac{m_{2k}}{(n-k)!(2k)!} z^k$ has only real roots for all n, where $m_{2k} = \int_0^1 t^{2k} F(t)dt$.

A Little History on Fourier Transforms with Only Real Zeros

Pólya (1918) suggested that determining the class of functions whose Fourier transforms have only real zeros would be a 'rather artificial question' if it were not for the Riemann Hypothesis. For $\Re(s) > 1$, the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. It has an analytic continuation, and the function

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire. The Riemann Hypothesis states that all the zeros of $\xi(s)$ satisfy $\Re(s) = 1/2$.

It is well known, see Titchmarsh, that

$$\Xi(z) = \xi(\frac{1}{2} + \mathrm{i}z) = \int_{-\infty}^{\infty} \phi(x) e^{\mathrm{i}zx} dx$$

where

$$\phi(x) = \sum_{n=1}^{\infty} \left(4n^4\pi^2 e^{9x/2} - 6n^2\pi e^{5x/2}\right) \exp(-n^2\pi e^{2x}).$$

In other words, the Riemann Hypothesis is true if and only if the Fourier transform $\Xi(z)$ has only real zeros.

Fourier Transforms with Only Real Zeros and Probability

Our method here is based on the paper of Pólya (1918) and we have to show (with a lot of work) the following general result.

Thm 3. For any probability measure μ defined on $[-1,1]$ with $\mu((-1,1)) > 0$, the zeros of the entire function

$$\cos(z) - \int_{-1}^{1} \cos(xz)\, d\nu(x)$$

are all real.

•Proper Discretization.

•Transformation back into trigonometric functions again.

•Domination of high frequency trigonometric functions

Discrete Setting

Consider Markov Chain transition matrix

$$P = \begin{bmatrix} t_1 & \frac{1}{2}+t_2 & t_3 & t_4 & \cdots & t_{n-1} & t_n \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ s_1 & s_2 & s_3 & \cdots & s_{n-2} & \frac{1}{2}+s_{n-1} & s_n \end{bmatrix}$$

where

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} s_i = \frac{1}{2}, \quad t_i, s_i \geq 0$$

•Can you tell that all eigenvalues of P are **real**?

•Can you tell that **exact** half of them are positive?

Related Problems

•W.V. Li and J. Peng (2008+), Brownian motion with holding and jump boundary,

•Jump-in Levy processes, W. Chu, W.V. Li and Y. Ren.

•Distribution of the maximum eigenvalue in the discrete setting.

•Comparisons with symmetrized (reversible) case.

•Markov vs Gaussian processes (Isomorphism, Gaussian free fields, etc);