Optimal Ventcel Graphs,Minimal Cost Spanning Trees and Asymptotic Probabilities

Chiang,Tzuu-Shuh   Chow,Yunshyong

Institute of Mathematics
Academia Sinica,Taipei,Taiwan

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Abstract and Basic Results

Let X_n^ϵ be an aperiodic irreducible Markov Chain on a finite set S with transition function $p^\epsilon = p_{i,j}\epsilon^{U(i,j)}(1 + \circ(1))$ where $0 \leq U(i,j) \leq \infty$ will be referred to as a cost function. We assume $p_{i,j} = 0$ if $U(i,j) = \infty$ and let μ^ϵ be the invariant distribution of X_n^ϵ. It is well-known that there are constants $h(i) \geq 0, \beta_i > 0$ such that

- $lim_{\epsilon\downarrow 0}\mu^\epsilon(i)/\epsilon^{h(i)} = \beta_i$ for any $i \in S$.
- Let $\underline{S} = \{i \in S, h(i) = 0\}$. (set of minimal states.) Then,

$$E^\epsilon T \approx \epsilon^{\delta_h} \quad \text{for small } \epsilon.$$

  Here T is the hitting time of $\underline{S}$ and the expectation is from the invariant distribution.
- $E^\epsilon T_i \approx \epsilon^{\delta_v}$ where i is any fixed state in $\underline{S}$.

The problem we wish to address is to use Ventcel graphs (minimal spanning trees) to describe the constants

$$h(i), \delta_h \text{ and } \delta_v.$$

- All these constants have been discussed before using the so called cycles..
- Conceptually easy through "cycles".
- Hard in real computation.
- X_n^ϵ does not converge to X_n^0. Hence a singular perturbation.

Motivation and Ising model as an example

Take the Ising model as an example. Let $u(i)$ be the energy of a state $i \in S$. At temperature T, the probability being at i is

$$P_T(i) \propto exp(-u(i)/T) \quad (P_T(i) = 1/Z \cdot exp(-u(i)/T))$$

where

$$Z = \Sigma_i exp(-u(i)/T).$$

In general, it is difficult to find the ground states of this enengy landscape. One way to find them using computers is the following.

Motivation Continued

- To run a Markov chain X_n^T on S with transition

$$P_{i,j}^T = p_{i,j} exp(-(u(j) - u(i))^+)/T, \quad j \in N(i).$$

  where $N(i)$ is a neighborhood system of u and usually contains very few states.
- $u(j) - u(i)$ is easy to compute. NOT $u(i)$.
- $P(X_n^T \in$ ground states$) \rightarrow 1$ for small T.
- Example. In the Ising Model, $u(i) = -\Sigma_{|b-b\prime|=1} j(b, b')i(b)i(b')$. Here b, b' are lattice points and i is a configuration. $i(b) = 1$ or -1.

Ventcel graphs

To be moer general,let $u(i,j)$ be a cost function on $S \times S$ and let $W \subseteq S$.

- **Definition.** A W-Ventsel graph is a function g from $S \backslash W$ to S with no cycles.
- **Definition.** For a W-graph $V(g) = \Sigma_{i \in S \backslash W} u(i, g(i))$ is called the cost of g.
- **Definition.** A W-graph g is called W-optimal if $V(g) \leq V(g')$ for any W-graph g'. We use $v(W)$ for the value $V(g)$ where g is W-optimal
- **Definition.** For any positive integer k, let $v(k) = min\{V(g), g \text{ is a W-graph with } |W| = k\}$.

Main Results

Our main results are as follows.

- (1) $h(i) = v(\{i\}) - v_1$
- (2) $\delta_v = v_1 - v_2$ and
- (3) $\delta_h = v_{k_0 - 1} - v_{k_0}$ where $k_0 = min\{k :$ there exists an k opitmal W-graph with $W \not\subset \underline{S}\}$.

Previous Results using cycles

- For a cost function $u(i,j)$, let $V(i) = min_{j \neq i} u(i,j)$.
- For two states $i, j \in S$, we say $i \geq j$ if there exists $i_0 = i, i_1, ...i_m = j$ such that $u(i_k, i_{k+1}) = v(i)$ for each k.
- i is minimal if $i \geq j$ implies $j \geq i$.
- Two different states i, j are said equivalent if $i \geq j$ and $j \geq i$ and i, j are minimal. Each equivalent class is called a "cycle".

Cycles continued

Let $S^0 = S, u^0 = u$ and $V^0 = V$. Let $S^1 =$ all cycles of S^0. Hence if $C^1 \in S^1$, then$C^1 = \{C_i^1\}_i$ where $C_i^1 \in S^0$. Let the depth of C^1 be defined as $d(C^1) = max\{V^0(C_i^1) : C_i^1 \in C^1\}$.

- **Definition.** $u^1(C^1, \bar{C}^1) = d(C^1) + min_{i,j}\{u^0(C_i^1, \bar{C}_j^1) - V^0(C_i^1)\}$.
- S^1, u^1 and V^1 forms a new cost function where $V^1(C^1) = minu^1(C^1, \bar{C}^1), \bar{C}^1 \neq C^1$.

Cycles continued

- The process will end at some N.
- For each state $i \in S$, there will be a sequence of cycles $i = C^0 \in C^1 \in C^2 \in ... \in C^N$. (The family tree of i.)

Cycles continued

The constants can be defined as follows.

- $\underline{S} = \{i \in S, h(i) = 0$ where $h(i) = \Sigma_n d^n(C^{n+1}) - V^n(C^n)\}$.
- $\delta_h = max\{V^k(C^k), C^k \in S^k, C^k \cap \bar{S} = \phi\}$.
- $\delta_v = max\{V^k(C^k), C^k \in S^k, i_0 \notin C^k\}$ where $i_0 \in \underline{S}$ is fixed.

References I

Chiang,Tzuu-Shuh, Chow,Yunshyong
Optimal Ventcel Graphs, minimal cost spanning trees and asymptotic probabilities.
Applicable Analysis and Discrete Mathematics, 1(2007),265–275.

Chiang,Tzuu-Shuh, Chow,Yunshyong
Asymptotic behavior of eigenvalues and random updating schemes.
Appl.Math.Optim., 28(1993),259–275.

Chen,H.C., Chow,Yunshyong
Cooperation in prisoner's dilemma games with local interaction and imitation.
preprint, (2005).