Introduction to Wasserstein Spaces

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The topic on the heat equation

$$\frac{\partial u_t}{\partial t} - \Delta u_t = 0 \qquad \text{on } \mathbb{R}^d$$

is classical; but the porous medium equation

$$\frac{\partial u_t}{\partial t} - (\Delta u_t^m) = 0 \qquad m \neq 1 \tag{0.1}$$

arises interests among probabiliste. More precisely, for $m > 1 - \frac{1}{d}$, $u_0 \geq 0$ such that $\int u_0 dx = 1$ and $\int |x|^2 u_0 dx < +\infty$, then the weak solution to (0.1) can be interpreted as the solution to the following ordinary differential equation (ODE)

$$\begin{cases} \frac{d\rho_t}{dt} = -\nabla\psi(\rho_t) \\ \rho_t|_{t=0} = u_0 dx \end{cases} \tag{0.2}$$

where $\rho_t \in \mathbb{P}_2(\mathbb{R}^d)$ and $\psi : \mathbb{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is a convex functional. A quite general theory says that for two initial data ρ_0^1 and ρ_0^2, then

$$t \mapsto W_2(\rho_t^1, \rho_t^2) \text{ is decreasing.}$$

In particular, (0.2) admits a unique solution. The purpose of this lecture is to understand the geometric structure of $\mathbb{P}_2(\mathbb{R}^d)$.

1 Wasserstein Space $(\mathbb{P}_2(\mathbb{R}^d), W_2)$

1.1 Wasserstein distance

Let

$$\mathbb{P}_2(\mathbb{R}^d) = \left\{ \mu \text{ is a probability measure on } \mathbb{R}^d; m_2(\mu) := \int_{\mathbb{R}^d} |x|^2 d\mu(x) < +\infty \right\}.$$

For μ, $\nu \in \mathbb{P}_2(\mathbb{R}^d)$, we define

$$W_2^2(\mu,\nu) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y) : \gamma \in \mathscr{C}(\mu,\nu) \right\},$$

where $\mathscr{C}(\mu,\nu) = \{\gamma \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^d) : (\pi_1)_*\gamma = \mu, (\pi_2)_*\gamma = \nu\}$, here $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection on the first component, while π_2 is on the second one. It is sometimes convenient to use another more probabilistic formulation:

$$W_2^2(\mu,\nu) = \inf\left\{\mathbb{E}(|X-Y|^2) : \text{law}(X) = \mu, \text{law}(Y) = \nu\right\}.$$

For μ, $\nu \in \mathbb{P}_2(\mathbb{R}^d)$, $W_2(\mu,\nu) < +\infty$, since

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) &\leq 2\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2 d\gamma(x,y) + \int_{\mathbb{R}^d\times\mathbb{R}^d} |y|^2 d\gamma(x,y)\right)\\ &= 2(m_2(\mu) + m_2(\nu)) < \infty\end{aligned}$$

Proposition 1.1 *There is a* $\gamma_0 \in \mathscr{C}(\mu,\nu)$ *such that*

$$W_2^2(\mu,\nu) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_0(x,y).$$

Proof. By the above remark, $W_2^2(\mu,\nu) = \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) : \gamma \in \mathscr{C}(\mu,\nu)\right\}$ is finite, therefore for each $n \geq 1$, there exists $\gamma_n \in \mathscr{C}(\mu,\nu)$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_n(x,y) \leq W_2^2(\mu,\nu) + \frac{1}{n}.$$

Let $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^d$ such that

$$\mu(K^c) + \nu(K^c) \leq \varepsilon.$$

Now $(K \times K)^c \subset (K^c \times \mathbb{R}^d) \cup (\mathbb{R}^d \times K^c)$,

$$\begin{aligned}\gamma_n((K\times K)^c) &\leq \gamma_n(K^c \times \mathbb{R}^d) + \gamma_n(\mathbb{R}^d \times K^c)\\ &= \mu(K^c) + \nu(K^c) < \varepsilon\end{aligned}$$

Therefore the family $\{\gamma_n; n \geq 1\}$ is tight. Up to a subsequence, γ_n converges to $\gamma \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then $\gamma \in \mathscr{C}(\mu,\nu)$, in fact for any $\varphi \in C_b(\mathbb{R}^d)$,

$$\int \varphi(x) d\mu(x) = \int \varphi(x) d\gamma_n(x,y) \to \int \varphi(x) d\gamma(x,y),$$

so $(\pi_1)_*\gamma = \mu$. In the same way, $(\pi_2)_*\gamma = \nu$.
Let $R > 0$. We have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (|x-y|^2 \wedge R) d\gamma_n(x,y) \leq W_2^2(\mu,\nu) + \frac{1}{n},$$

letting $n \to \infty$ gives

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (|x-y|^2 \wedge R) d\gamma(x,y) \leq W_2^2(\mu,\nu),$$

Letting $\mathbb{R} \to +\infty$, we get the results. □

In what follows, we denote by

$$\begin{aligned}\mathscr{C}_0(\mu,\nu) &= \{\text{optimal coupling of } \mu \text{ and } \nu\}\\ &= \left\{\gamma_0 \in \mathscr{C}(\mu,\nu):\ W_2^2(\mu,\nu) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_0(x,y)\right\}.\end{aligned}$$

In fact, $\mathscr{C}_0(\mu,\nu)$ is a convex subset of $\mathscr{C}(\mu,\nu)$.

Kantorovich problem: when $\mathscr{C}_0(\mu,\nu)$ has only one element?
Minge problem: when $\gamma_0 = (I\times T)_*\mu$? How is about the regularity of T?

Roughly speaking, the Wasserstein distance is realized for two highly correlated random variables (X,Y).

Proposition 1.2 *W_2 is the distance on $\mathbb{P}_2(\mathbb{R}^d)$.*

Proof.(i) Let $T: x \to (x,x)$ and $\gamma = T_*\mu$. Then $\gamma \in \mathscr{C}(\mu,\mu)$ and

$$W_2^2(\mu,\nu) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\mu(x) = 0.$$

Conversely, if $W_2(\mu,\nu)=0$, take a $\gamma_0 \in \mathscr{C}_0(\mu,\nu)$ such that $\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) = 0$. It follows that γ is supported by the diagonal; so that

$$\int \varphi(x)d\mu(x) = \int \varphi(x)d\gamma(x,y) = \int \varphi(y)d\gamma(x,y) = \int \varphi(y)d\nu(y)$$

Hence $\mu=\nu$.
(ii) Consider the map $T:(x,y)\to(y,x)$. For any $\gamma\in\mathscr{C}_0(\mu,\nu)$ and $\varphi\in C_b(\mathbb{R}^d)$, define $\hat\gamma = T_*\gamma$ and $\tilde\varphi(x,y)=\varphi(x)$. Then

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x)d\hat\gamma(x,y) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \tilde\varphi(x,y)d\hat\gamma(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \tilde\varphi(T(x,y))d\gamma(x,y)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y)d\gamma(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y)d\nu(y)\end{aligned}$$

so $(\pi_1)_*\hat\gamma = \nu$. In the same way, $(\pi_2)_*\hat\gamma=\mu$. Therefore $\hat\gamma\in\mathscr{C}(\nu,\mu)$ and

$$W_2^2(\nu,\mu) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\hat\gamma(x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) = W_2^2(\mu,\nu)$$

Changing the roles, we get the equality.
(iii) Let $\mu_1,\mu_2,\mu_3\in\mathbb{P}(\mathbb{R}^d)$. Let $\gamma_1\in\mathscr{C}_0(\mu_1,\mu_2)$, $\gamma_2\in\mathscr{C}_0(\mu_2,\mu_3)$. Then by the result below, $\exists\lambda\in\mathbb{P}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ such that

$$(\pi_1,\pi_2)_*\lambda = \gamma_1,\quad (\pi_2,\pi_3)_*\lambda=\gamma_2.$$

Consider $\gamma = (\pi_1,\pi_3)_*\lambda$. Then

$$\begin{aligned}\int \varphi(x)d\gamma(x,z) &= \int \varphi(x)d\lambda(x,y,z) = \int \varphi(x)d\gamma_1(x,y) = \int \varphi d\mu_1,\\ \int \varphi(z)d\gamma(x,z) &= \int \varphi(z)d\lambda(x,y,z) = \int \varphi(z)d\gamma_2(y,z) = \int \varphi d\mu_3.\end{aligned}$$

Thus $\gamma \in \mathscr{C}(\mu_1, \mu_3)$, we have

$$\begin{aligned} W_2(\mu_1,\mu_3) &\leq \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1-x_3|^2 d\lambda(x_1,x_2,x_3) \right)^{\frac{1}{2}} \\ &\leq \| d(x_1,x_2) \|_{L^2(\lambda)} + \| d(x_2,x_3) \|_{L^2(\lambda)} \\ &= \| d(x_1,x_2) \|_{L^2(\gamma_1)} + \| d(x_2,x_3) \|_{L^2(\gamma_2)} \\ &= W_2(\mu_1,\mu_2) + W_2(\mu_2,\mu_3), \end{aligned}$$

here $d(x,y) = |x-y|$. □

Theorem 1.1 *Let E_1, E_2, E_3 be Polish space. Let $\gamma^{12} \in \mathbb{P}(E_1 \times E_2)$, $\gamma^{23} \in \mathbb{P}(E_2 \times E_3)$. Suppose that $\nu := (\pi_2)_*\gamma^{12} = (\pi_1)_*\gamma^{23}$. Then there exists a $\lambda \in \mathbb{P}(E_1 \times E_2 \times E_3)$, such that $(\pi_1, \pi_2)_*\lambda = \gamma^{12}$, $(\pi_2, \pi_3)_*\lambda = \gamma^{23}$.*

Proof. We have $\pi_2 : E_1 \times E_2 \to E_2$ and $(\pi_2)_*\gamma^{12} = \nu$. Let $\gamma_y^{12}(dx)$ be the conditional probability on E_1 of γ^{12} given $\{\pi_2 = y\}$. Note that $\gamma_y^{12} \in \mathbb{P}(E_1)$ is defined only for γ-a.s., y. In term of probability, γ^{12} is the joint law of a couple of random variables (X, Y), ν is the law of Y and γ_y^{12} is the conditional law of X given $\{Y = y\}$. That is,

$$\int_{E_1\times E_2} f(x,y)d\gamma^{12}(x,y) = \int_{E_2} \left(\int_{E_1} f(x,y)\gamma_y^{12}(dx) \right) d\nu(y). \tag{1.1}$$

For γ^{23}, we write, in the same way,

$$\int_{E_2\times E_3} \varphi(y,z)d\gamma^{23}(y,z) = \int_{E_2} \left(\int_{E_3} \varphi(y,z)\gamma_y^{23}(dz) \right) d\nu(y). \tag{1.2}$$

Define a measure $\lambda \in \mathbb{P}(E_1 \times E_2 \times E_3)$ by

$$\int_{E_1\times E_2\times E_3} \varphi(x,y,z)d\lambda(x,y,z) = \int_{E_2} \left(\int_{E_1\times E_3} \varphi(x,y,z)\gamma_y^{12}(dx)\gamma_y^{23}(dz) \right) d\nu(y). \tag{1.3}$$

If $\varphi(x,y,z) = \varphi(x,y)$,

$$\int_{E_1\times E_3} \varphi(x,y)\gamma_y^{12}(dx)\gamma_y^{23}(dz) = \int_{E_1} \varphi(x,y)\gamma_y^{12}(dx). \tag{1.4}$$

This implies that $(\pi_1, \pi_2)_*\lambda = \gamma^{12}$. In the same way, we see that $(\pi_2, \pi_3)_*\lambda = \gamma^{23}$. □

Theorem 1.2 *Let μ_n, $\mu \in \mathbb{P}_2(\mathbb{R}^d)$, then $\lim_{n\to\infty} W_2(\mu_n, \mu) = 0$ if and only if*
i) $\mu_n \to \mu$ weakly;
ii) (μ_n) has uniformly integrable 2-moment, i.e.,

$$\lim_{R\to+\infty} \sup_n \left(\int_{|x|\geq R} |x|^2 d\mu_n(x) \right) = 0.$$

Proof. We first prove the converse part. By Skorohod representation theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of random variables X_n and X such that

$$\mathrm{law}(X_n) = \mu_n, \mathrm{law}(X) = \mu,$$

and X_n converges to X almost surely. The condition ii) implies that $\{|X_n - X|^2; n \geq 1\}$ is uniformly integrable; thus, we have

$$W_2^2(\mu_n, \mu) \leq \mathbb{E}(|X_n - X|^2) \to 0.$$

Now suppose $W_2^2(\mu_n, \mu) \to 0$, we prove first the weak convergence of μ_n to μ.
(1) If φ is 1-Lipschitz, i.e.,

$$|\varphi(x) - \varphi(y)| \leq |x - y|,$$

then

$$\Big|\int \varphi d\mu - \int \varphi d\nu\Big| \leq \int |x-y| d\gamma(x,y) \leq \Big(\int |x-y|^2 d\gamma(x,y)\Big)^{\frac{1}{2}}$$

Taking the infinimum over $\gamma \in \mathscr{C}(\mu, \nu)$ on the right side, we get

$$\Big|\int \varphi d\mu - \int \varphi d\nu\Big| \leq W_2(\mu, \nu).$$

Therefore for any 1-Lipschitz function φ,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi d\mu_n = \int_{\mathbb{R}^d} \varphi d\mu. \tag{1.5}$$

By considering $\frac{\varphi}{\|\varphi\|_{Lip}}$, (1.5) holds for any lipschtiz function, in particular for $\varphi \in C_c^1(\mathbb{R}^d)$.
(2) Let $\varphi \in C_b(\mathbb{R}^d)$, consider the cut-off function $\chi_R \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \chi_R \leq 1$ and

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R. \end{cases}$$

Then $\varphi_R := \varphi \cdot \chi_R \in C_c(\mathbb{R}^d)$. We have

$$\begin{aligned} \Big|\int \varphi d\mu_n - \int \varphi_R d\mu_n\Big| &\leq \int |\varphi|(1 - \varphi_R) d\mu_n \\ &\leq \parallel \varphi \parallel_\infty \cdot \mu_n\{|x| > R\} \leq \parallel \varphi \parallel_\infty \cdot \frac{m_2(\mu_n)}{R^2}. \end{aligned}$$

Let $\varepsilon > 0$, $\exists R_0$ such that

$$\mu_n\{|x| > R\} \leq \frac{\varepsilon}{\parallel \varphi \parallel_\infty}, \quad \forall n \geq 1$$

so

$$\sup_n \Big|\int \varphi d\mu_n - \int \varphi_R d\mu_n\Big| \leq \varepsilon, \ \forall R \geq R_0. \tag{1.6}$$

Now take $\psi \in C_c^1(\mathbb{R}^d)$ such that $\parallel \varphi_R - \psi \parallel_\infty \leq \varepsilon$, we have

$$\sup_n \Big|\int \varphi_R d\mu_n - \int \psi d\mu_n\Big| \leq \varepsilon,$$

but

$$\Big|\int \psi d\mu_n - \int \psi d\mu\Big| \le \varepsilon \ \text{ for } \ n \ge n_0. \tag{1.7}$$

Combining (1.6), (1.7), we get

$$\begin{aligned}\Big|\int \varphi d\mu_n - \int \varphi d\mu\Big| \le &\Big|\int \varphi d\mu_n - \int \varphi_R d\mu_n\Big| + \Big|\int \varphi_R d\mu_n - \int \psi d\mu_n\Big| \\ &+ \Big|\int \psi d\mu_n - \int \psi d\mu\Big| + \Big|\int \psi d\mu - \int \varphi_R d\mu\Big| \\ &+ \Big|\int \varphi_R d\mu - \int \varphi d\mu\Big| \le 5\varepsilon \ \text{ for } \ n \ge n_0.\end{aligned}$$

Now we prove ii). For $\varepsilon > 0$, $\xi \in \mathbb{R}$, we see that

$$(1+\xi)^2 - (1+\varepsilon)\xi^2 = -\varepsilon(\xi - \frac{1}{\varepsilon})^2 + (1 + \frac{1}{\varepsilon}) \le 1 + \frac{1}{\varepsilon} := C_\varepsilon,$$

so for $a, b \in \mathbb{R}$,

$$(a+b)^2 \le (1+\varepsilon)a^2 + C_\varepsilon b^2.$$

Take $\gamma_n \in \mathscr{C}_0(\mu_n, \mu)$, we have

$$\int_{\mathbb{R}^d} |x|^2 d\mu_n = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 d\gamma_n \le (1+\varepsilon) \int_{\mathbb{R}^d} |y|^2 d\mu + C_\varepsilon W_2^2(\mu_n, \mu).$$

It follows that

$$\limsup_{n \to +\infty} \int |x|^2 d\mu_n \le (1+\varepsilon) \int |x|^2 d\mu.$$

Letting $\varepsilon \downarrow 0$, we get the following

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^d} |x|^2 d\mu_n(x) \le \int_{\mathbb{R}^d} |x|^2 d\mu(x). \tag{1.8}$$

Let $\delta > 0$, $\exists\, n \ge n_0$ such that when $n \ge n_0$,

$$\int_{\mathbb{R}^d} |x|^2 d\mu_n(x) \le \int_{\mathbb{R}^d} |x|^2 d\mu + \delta. \tag{1.9}$$

On the other hand, for $R > 0$ given, $x \mapsto 1_{\{|x|<R\}}(|x|^2) \wedge M$ is lower semi-continuous, then

$$\liminf_{n \to \infty} \int_{|x|<R} |x|^2 \wedge M d\mu_n(x) \ge \int_{|x|<R} |x|^2 \wedge M d\mu(x)$$

or

$$\liminf_{n \to \infty} \int_{|x|<R} |x|^2 d\mu_n(x) \ge \int_{|x|<R} |x|^2 \wedge M d\mu(x).$$

Letting $M \uparrow +\infty$ leads to

$$\liminf_{n \to \infty} \int_{|x|<R} |x|^2 d\mu_n(x) \ge \int_{|x|<R} |x|^2 d\mu(x).$$

Or for $n \geq n_0$,

$$\int_{|x|<R} |x|^2 d\mu_n(x) \geq \int_{|x|<R} |x|^2 d\mu(x) - \varepsilon. \tag{1.10}$$

It follows that

$$\begin{aligned}\int_{|x|\geq R} |x|^2 d\mu_n(x) &= \int_{\mathbb{R}^d} |x|^2 d\mu_n(x) - \int_{|x|<R} |x|^2 d\mu_n(x) \\ &\leq \int_{\mathbb{R}^d} |x|^2 d\mu(x) - \int_{|x|<R} |x|^2 d\mu(x) + 2\varepsilon \\ &= \int_{|x|\geq R} |x|^2 d\mu(x) + 2\varepsilon.\end{aligned}$$

Thus

$$\sup_{n\geq n_0} \int_{|x|\geq R} |x|^2 d\mu_n(x) \leq \int_{|x|\geq R} |x|^2 d\mu(x) + 2\varepsilon$$

. Then we have

$$\lim_{R\to\infty} \sup_{n\geq 1} \int_{|x|\geq R} |x|^2 d\mu_n \leq 2\varepsilon.$$

Let $\varepsilon \downarrow 0$, we get the desired result. □

Theorem 1.3 *The space $(\mathbb{P}_2(\mathbb{R}^d), W_2)$ is complete.*

Proof. Let $\{\mu_n; n \geq 1\}$ be a Cauchy sequence in $(\mathbb{P}_2(\mathbb{R}^d), W_2)$. Then for $\varepsilon > 0$, $\exists\, n_0$ such that

$$W_2(\mu_n, \mu_m) \leq \varepsilon, \qquad \text{for} \quad n, m \geq n_0. \tag{1.11}$$

Note that

$$W_2(\mu_n, \mu_1) \leq W_2(\mu_n, \mu_{n_0}) + W_2(\mu_{n_0}, \mu_1),$$

so

$$\sup_n W_2(\mu_n, \mu_1) < \infty$$

which implies that

$$m := \sup_n m_2(\mu_n) < +\infty.$$

Therefore the family $\{\mu_n; n \geq 1\}$ is tight, since

$$\mu_n(\{|x| > R\}) \leq \frac{1}{R^2} \int |x|^2 d\mu_n \leq \frac{m}{R^2}.$$

There exists a subsequence $\{\mu_{n_k}\}$ such that $\mu = \lim_{k\to\infty} \mu_{n_k}$ weakly. Let $\gamma_{n,n_k} \in \mathscr{C}_0(\mu_n, \mu_{n_k})$. Then, as in the proof of Proposition 1.1, the family $\{\gamma_{n,n_k;k\geq 1}\}$ is tight. Up to a subsequence of n_k, $\gamma_{n,n_k} \to \gamma_{n,\infty}$ weakly. Now for $\varphi \in C_b(\mathbb{R}^d)$,

$$\int \varphi d\mu = \lim_{k\to\infty} \int \varphi d\mu_{n_k} = \lim_{k\to\infty} \int \varphi d\gamma_{n,n_k} = \int \varphi d\gamma_{n,\infty}$$

so $(\pi_2)_*\gamma_{n,\infty} = \mu$ and $\gamma_{n,\infty} \in \mathscr{C}(\mu_n, \mu)$.
Let $R > 0$.

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \wedge R \; d\gamma_{n,\infty} = \lim_{k\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \wedge R \; d\gamma_{n,n_k},$$

but for $n \geq n_0$, k big enough

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \wedge R \; d\gamma_{n,n_k} &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_{n,n_k} \\ &= W_2^2(\mu_n, \mu_{n_k}) \leq \varepsilon^2,\end{aligned}$$

so

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \wedge R \; d\gamma_{n,\infty} \leq \varepsilon^2.$$

Letting $R \uparrow \infty$, gives

$$W_2(\mu_n, \mu) \leq \varepsilon \quad \text{for} \quad n \geq n_0$$

□

2 Geometric properties

Let $\mu, \nu \in \mathbb{P}_2(\mathbb{R}^d)$. Pick $\gamma \in \mathscr{C}_o(\mu, \nu)$. Define

$$\mu_t := ((1-t)\pi_1 + t\pi_2)_*\gamma, \qquad t \in [0,1]$$

that is

$$\int_{\mathbb{R}^d} \varphi d\mu_t = \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi((1-t)x + ty) d\gamma(x,y).$$

Then, $\mu_0 = (\pi_1)_*\gamma = \mu$ and $\mu_1 = (\pi_2)_*\gamma = \nu$. Note that it is easy to see that $\mu_t \in \mathbb{P}_2(\mathbb{R}^d)$.

Proposition 2.1 *We have for* $0 \leq s < t \leq 1$, $W_2(\mu_s, \mu_t) = (t-s)W_2(\mu, \nu)$.

Proof. Define $\gamma_{s,t} \in \mathscr{C}(\mu_s, \mu_t)$ by

$$\gamma_{s,t} = ((1-s)\pi_1 + s\pi_2, (1-t)\pi_1 + t\pi_2)_*\gamma, \quad \gamma \in \mathscr{C}_o(\mu, \nu), \tag{2.12}$$

or

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y) d\gamma_{s,t} = \int_{\mathbb{R}^d\times\mathbb{R}^d} f((1-s)x + sy, (1-t)x + ty) d\gamma(x,y). \tag{2.13}$$

Then

$$\begin{aligned}W_2^2(\mu_s, \mu_t) &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_{s,t} \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} |(1-s)x + sy - ((1-t)x + ty)|^2 d\gamma(x,y) \\ &= (t-s)^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) \\ &= (t-s)^2 W_2^2(\mu, \nu). \end{aligned} \tag{2.14}$$

This implies that

$$W_2(\mu_s, \mu_t) \le (t-s) W_2(\mu, \nu).$$

If for some $s_0 < t_0$, it holds

$$W_2(\mu_{s_0}, \mu_{t_0}) < (t_0 - s_0) W_2(\mu, \nu),$$

then

$$\begin{aligned} W_2(\mu,\nu) &= W_2(\mu_0,\mu_1) \le W_2(\mu_0,\mu_{s_0}) + W_2(\mu_{s_0},\mu_{t_0}) + W_2^2(\mu_{t_0},\mu_1) \\ &< s_0 W_2(\mu,\nu) + (t_0 - s_0) W_2(\mu,\nu) + (1-t_0) W_2(\mu,\nu) = W_2(\mu,\nu). \end{aligned} \tag{2.15}$$

This is a contradiction. Therefore

$$W_2(\mu_s, \mu_t) = (t-s) W_2(\mu, \nu).$$

□

Note that the above proposition implies that for $0 \le t_1 < t_2 < t_3 \le 1$,

$$W_2(\mu_{t_1}, \mu_{t_3}) = W_2(\mu_{t_1}, \mu_{t_2}) + W_2(\mu_{t_2}, \mu_{t_3}).$$

Definition 2.1 *Let $(\mu_t)_{t\in[0,1]}$ be a curve in $\mathbb{P}_2(\mathbb{R}^d)$. We say that it is absolutely continuous in $\mathcal{AC}_2$ if $W_2(\mu_s, \mu_t) \le \int_s^t m(\tau) d\tau, \quad s < t, \; m \in L^2([0,1])$.*

Example 2.1 *Let $Z : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 vector field with bounded derivative. The differential equation*

$$\frac{dX_t}{dt} = Z(X_t), \quad X_t|_{t=0} = x \tag{2.16}$$

defines a flow of diffeomorphism $U_t : \mathbb{R}^d \to \mathbb{R}^d$ by $U_t(x) = X_t$ with $X_t|_{t=0} = x$.

Let $\mu_0 \in \mathbb{P}_2(\mathbb{R}^d)$ and consider $\mu_t = (U_t)_* \mu_0$. Then $\mu_t \in \mathbb{P}_2(\mathbb{R}^d)$. Let $s < t$. Define $\gamma_{s,t} \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x,y) d\gamma_{s,t}(x,y) = \int_{\mathbb{R}^d} \varphi(U_s, U_t) d\mu_0. \tag{2.17}$$

Then $\gamma_{s,t} \in \mathscr{C}(\mu_s, \mu_t)$. We have

$$\begin{aligned} W_2^2(\mu_s,\mu_t) &\le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma_{s,t}(x,y) \\ &= \int_{\mathbb{R}^d} |U_s(x) - U_t(x)|^2 d\mu_0(x). \end{aligned} \tag{2.18}$$

But $|U_s(x) - U_t(x)| = \int_s^t |Z(U_\tau)| d\tau$. Then

$$\begin{aligned} W_2^2(\mu_s,\mu_t) &\le \left\| \int_s^t |Z(U_\tau)| d\tau \right\|_{L^2(\mu_0)} \\ &\le \int_s^t \|Z(U_\tau)\|_{L^2(\mu_0)} \, d\tau = \int_s^t m(\tau) d\tau. \end{aligned} \tag{2.19}$$

Note that $|Z(x) - Z(y)| \leq c|x - y|$, implying that $|Z(x)| \leq c(1 + |x|)$. Then since $U_t(x) = x + \int_0^t Z(U_s(x))ds$, we have

$$|U_t(x)| \leq |x| + c\int_0^t (1 + |U_s(x)|)ds = |x| + c + c\int_0^t |U_s(x)|ds. \tag{2.20}$$

Gronwall lemma implies that

$$|U_t(x)| \leq (|x| + c)e^{ct} \leq c_1(1 + |x|). \tag{2.21}$$

Then $|U_t(x)|^2 \leq c_2^2(1 + |x|)^2 \leq 2c_2^2(1 + |x|^2)$ and

$$\begin{aligned}\int_0^1 m(\tau)^2 d\tau &= \int_0^1 \int_{\mathbb{R}^d} |Z(U_\tau)|^2 d\mu_0(x)d\tau \\ &\leq 2c_2^2 \int_{\mathbb{R}^d} (1 + |x|^2)d\mu_0(x) = 2c_2^2(1 + m_2(\mu_0)) < \infty.\end{aligned}$$

□

Theorem 2.1 *Let $(\mu_t)_{t\in[0,1]}$ be an absolutely continuous curve in $\mathcal{AC}_2$. Then there exists a Borel vector field $Z : (t, x) \mapsto Z_t(x) \in \mathbb{R}^d$ such that*

(i) $Z_t \in L^2(\mathbb{R}^d, \mu_t)$, $\| Z_t \|_{L^2(\mu_t)} \leq m(t)$ *a.s.* $t \in (0, 1)$;

(ii) *the continuity equation*

$$\frac{\partial \mu_t}{\partial t} + \nabla \cdot (Z_t \mu_t) = 0,$$

holds in the sense that

(iii)

$$\int_{[0,1]\times\mathbb{R}^d} (\alpha'(t)\varphi(x) + \alpha(t) < Z_t(x), \nabla\varphi(x) >)d\mu_t dt = 0 \tag{2.22}$$

for $\alpha \in C_c^\infty((0, 1))$, $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Proof. For $\varphi \in C_b(\mathbb{R})$, denote $\mu_t(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu_t$. Then for $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$|\mu_t(\varphi) - \mu_s(\varphi)| = \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} (\varphi(y) - \varphi(x))d\gamma_{s,t} \right| \leq ||\nabla\varphi||_\infty \cdot W_2(\mu_s, \mu_t),$$

where $\gamma_{s,t} \in \mathscr{C}_0(\mu_s, \mu_t)$. The function $t \mapsto \mu_t(\varphi)$ is absolutely continuous .
Let $s \in (0, 1)$ be given and $\eta > 0$ small enough. We consider $\gamma_\eta \in \mathscr{C}_0(\mu_s, \mu_{s+\eta})$. For $x, y \in \mathbb{R}^d$, and $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\varphi(y) - \varphi(x) = \int_0^1 < (\nabla\varphi)(ty + (1 - t)x), y - x > dt. \tag{2.23}$$

Set

$$H(x, y) = \int_0^1 (\nabla\varphi)(ty + (1 - t)x)dt \in \mathbb{R}^d.$$

Then

$$\int_{\mathbb{R}^d} \varphi d\mu_{s+\eta} - \int_{\mathbb{R}^d} \varphi d\mu_s = \int_{\mathbb{R}^d\times\mathbb{R}^d} (\varphi(y) - \phi(x)) d\gamma_\eta$$
$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} < H(x,y), y-x > d\gamma_\eta(x,y). \tag{2.24}$$

Then

$$\frac{1}{\eta}\left|\int_{\mathbb{R}^d} \varphi d\mu_{s+\eta} - \int_{\mathbb{R}^d} \varphi d\mu_s\right| \leq \frac{1}{\eta} W_2(\mu_s, \mu_{s+\eta}) \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |H(x,y)|^2 d\gamma_\eta\right)^2. \tag{2.25}$$

Take a sequence η_n such that

$$\lim_{n\to\infty} \frac{1}{\eta_n}\left|\int_{\mathbb{R}^d} \varphi d\mu_{s+\eta_n} - \int_{\mathbb{R}^d} \varphi d\mu_s\right| = \limsup_{\eta\downarrow 0} \frac{1}{\eta}\left|\int_{\mathbb{R}^d} \varphi d\mu_{s+\eta} - \int_{\mathbb{R}^d} \varphi d\mu_s\right|.$$

Since $W_2(\mu_s, \mu_{s+\eta_n}) \to 0$, we have by theorem 1.3 that $\mu_{s+\eta_n}$ converges weakly to μ_s as $n \to \infty$. Therefore the family $\{\gamma_{\eta_n}, n \geq 1\}$ is tight. Up to a subsequence, $\gamma_{\eta_n} \to \hat{\gamma}$ weakly for some $\gamma \in \mathcal{C}(\mu_s, \mu_s)$. We have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\hat{\gamma}(x,y) \leq \lim_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma_n(x,y) = \lim_{n\to\infty} W_2^2(\mu_s, \mu_{s+\eta_n}) = 0.$$

It follows that $\hat{\gamma}$ is supported by the diagonal

$$D = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}.$$

We have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} |H(x,y)|^2 d\gamma_{\eta_n} = \int_{\mathbb{R}^d\times\mathbb{R}^d} |H(x,y)|^2 d\hat{\gamma} = \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 d\mu_s(x).$$

Therefore for a.s. $s \in (0,1)$

$$\limsup_{\eta\downarrow 0} \frac{1}{\eta}\left|\int_{\mathbb{R}^d} \varphi d\mu_{s+\eta} - \int_{\mathbb{R}^d} \varphi d\mu_s\right| \leq m(s)\|\nabla\varphi\|_{L^2(\mu_s)} \tag{2.26}$$

since $\lim_{\eta\downarrow 0} \frac{1}{\eta}\int_s^{s+\eta} m(\tau)d\tau = m(s)$ for a.s. $s \in (0,1)$.
Now take $\delta > 0$ small enough such that

$$supp(\alpha) + (-\delta, \delta) \subset (0,1).$$

Then for $0 < \eta < \delta$,

$$\int_0^1 \int_{\mathbb{R}^d} \alpha(s)\varphi(x) d\mu_{s+\eta}(x) ds = \int_0^1 \int_{\mathbb{R}^d} \alpha(s-\eta)\varphi(x) d\mu_s(x) ds$$

and

$$I_\eta := \int_0^1 \frac{1}{\eta}\left[\int_{\mathbb{R}^d} \alpha(s)\varphi(x) d\mu_s(x) - \int_{\mathbb{R}^d} \alpha(s)\varphi(x) d\mu_{s+\eta}(x)\right] ds$$
$$= \int_0^1 \int_{\mathbb{R}^d} \frac{\alpha(s) - \alpha(s-\eta)}{\eta} \varphi(x) d\mu_s ds. \tag{2.27}$$

Then

$$\lim_{\eta\downarrow 0} I_\eta = \int_0^1 \int_{\mathbb{R}^d} \alpha'(s)\varphi(x)d\mu_s(x)ds.$$

Now according to (2.26)

$$\begin{aligned}\lim_{\eta\downarrow 0} |I_\eta| &\leq \int_0^1 m(s)|\alpha(s)| \parallel \nabla\varphi \parallel_{L^2(\mu_s)} ds \\ &= \int_0^1 m(s) \parallel \alpha(s)\nabla\varphi \parallel_{L^2(\mu_s)} ds \end{aligned} \tag{2.28}$$

or

$$\left|\int_0^1 \int_{\mathbb{R}^d} \alpha'(s)\varphi(x)d\mu_s(x)ds\right| \leq \sqrt{\int_0^1 m^2(s)ds} \left(\int_0^1 \int_{\mathbb{R}^d} |\alpha(s)\nabla\varphi(x)|^2 d\mu_s(x)ds\right)^{\frac{1}{2}}. \tag{2.29}$$

Let $\mathbb{P}_\mu$ be the probability measure on $[0,1]\times\mathbb{R}^d$ defined by

$$\int_{[0,1]\times\mathbb{R}^d} \psi(s,x)d\mathbb{P}_\mu(s,x) = \int_0^1 \int_{\mathbb{R}^d} \psi(s,x)d\mu_s(x)ds.$$

Introduce the vector space

$$V = \left\{ \sum_{i=1}^n \alpha_i(s)\nabla\varphi_i(x) : \alpha_i \in C_c^\infty((0,1)), \varphi_i \in C_c^\infty(\mathbb{R}^d), n = 1,2,\cdots \right\}$$

and $\bar{V}$ the closure of V under $L^2(\mathbb{P}_\mu)$. Define for

$$A = \sum_{i=1}^n \alpha_i(s)\nabla\varphi_i(x) \in V,$$

$$L(A) = -\sum_i \int_0^1 \int_{\mathbb{R}^d} \alpha_i'\varphi_i(x)d\mu_s(x)ds.$$

Note that due to the linearity of (2.27), the inequality (2.29) holds for A :

$$|L(A)| \leq \sqrt{\int_0^1 m(s)ds}\ \ \|A\|_{L^2(\mathbb{P}_\mu)}. \tag{2.30}$$

It follows that $L(A)$ is well defined, that is, if A admits another expression $A = \sum_{j=1}^m \beta_j(s)\nabla\tilde{\varphi}_j(x)$, then

$$\sum_{i=1}^n \alpha_i\nabla\varphi_i - \sum_{j=1}^m \beta_j(s)\nabla\tilde{\varphi}_j(x) = 0.$$

Therefore by (2.30)

$$0 = -\sum_i \int_0^1 \int_{\mathbb{R}^d} \alpha_i'(s)\varphi_i(x)d\mu_s(x)ds + \sum_j \int_0^1 \int_{\mathbb{R}^d} \beta_j'(s)\nabla\tilde{\varphi}_j(x)d\mu_s(x)ds.$$

$L(A)$ is independent of the expression. Again by (2.30), L is a bounded linear operator. Then there exists $Z \in \bar{V}$ such that

$$L(A) = \int_0^1 \int_{\mathbb{R}^d} < A(s,x), Z_s(x) > d\mu_s ds.$$

Taking $A = \alpha\nabla\varphi$, we have

$$-\int_0^1 \int_{\mathbb{R}^d} \alpha'(s)\varphi d\mu_s ds = \int_0^1 \int_{\mathbb{R}^d} \alpha(s) < \nabla\varphi(x), Z_s(x) > d\mu_(s)ds$$

so the continuity equation (2.22) holds. □

We define

$$\begin{aligned} T_\mu = & \text{ closure in } L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu) \text{ of } \{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\} \\ = & \text{ called tangent space of } \mathbb{P}_2(\mathbb{R}^d) \text{ at } \mu. \end{aligned} \tag{2.31}$$

Proposition 2.2 *Let Z be given in Theorem 2.1. Then for a.s. $t \in (0,1)$, $Z_t \in T_{\mu_t}$ and the solution to the continuity equation (2.16) satisfying this property is unique.*

Proof. Let $A_n \in V$ such that $\|z - A_n\|_{L^2(\mathbb{P}_\mu)} \to 0$, or

$$\lim_{n\to\infty} \int_0^1 \left(\int_{\mathbb{R}^d} |Z_t(x) - A_n(t,x)|^2 d\mu_t(x) \right) dt = 0.$$

Up to a subsequence, for a.s. $t \in (0,1)$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |Z_t(x) - A_n(t,x)|^2 d\mu_t(x) = 0.$$

This means that $Z_t \in T_{\mu_t}$. Now let $\hat{Z}$ be another solution to the continuity equation such that $\hat{Z}_t \in T_{\mu_t}$ for a.s. $t \in (0,1)$. Then we have

$$\int_0^1 \alpha(t) \left(\int_{\mathbb{R}^d} < Z_t(x) - \hat{Z}_t(x), \nabla\varphi > d\mu_t \right) dt = 0, \ \forall \varphi \in C_c^\infty(\mathbb{R}^d), \alpha \in C_c^\infty((0,1)).$$

It follows that there exists a full measure subset $L_\varphi \in (0,1)$ such that

$$\int_{\mathbb{R}^d} < Z_t(x) - \hat{Z}_t(x), \nabla\varphi(x) > d\mu_t = 0 \text{ for } t \in L_\varphi. \tag{2.32}$$

Let D be a dense countable subset of $C_c^\infty(\mathbb{R}^d)$. Set $L = \bigcap_{\varphi\in D} L_\varphi$. Pick $(\varphi_n) \in D$ such that

$$\|\nabla\varphi_n - \nabla\varphi\|_\infty \to 0$$

as $n \to \infty$. We have for $t \in L$, and $n \geq 1$,

$$\int_{\mathbb{R}^d} < Z_t(x) - \hat{Z}_t(x), \nabla\varphi_n(x) > d\mu_t = 0.$$

Letting $n \uparrow \infty$ gives

$$\int_{\mathbb{R}^d} < Z_t(x) - \hat{Z}_t(x), \nabla\varphi(x) > d\mu_t(x) = 0.$$

Therefore, $Z_t = \hat{Z}_t$ μ_t-a.s. □

Definition 2.2 *We say that Z_t is the derivative process of μ_t in the sense of Otto-Ambrosio-Savare and denote*

$$Z_t = \frac{d^o \mu_t}{dt} \in T_{\mu_t}.$$

Theorem 2.2 *The Wasserstein distance is a Riemannian distance:*

$$W_2^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \left\| \frac{d^o \mu_t}{dt} \right\|_{T_{\mu_t}}^2 dt; \mu_t \in \mathcal{AC}_2 \text{ connects } \mu_0 \text{ and } \mu_1 \right\}.$$

Proof. Let μ_0 and μ_1 be given. Consider the geodesic curve

$$\mu_t = ((1-t)\pi_1 + t\pi_2)_* \gamma, \quad \gamma \in \mathscr{C}_0(\mu_0, \mu_1)$$

Then by Proposition 2.1, μ_t is in $\mathcal{AC}_2$ with $m(s) = W_2(\mu_0, \mu_1)$. Now by the proof of Theorem 2.1

$$\| Z \|_{L^2(\mathbb{P}_\mu)} \leq W_2(\mu_0, \mu_1),$$

which implies that

$$\inf \left\{ \int_0^1 \left\| \frac{d^o \mu_t}{dt} \right\|_{T_{\mu_t}}^2 dt; \mu_t \in \mathcal{AC}_2 \text{ connects } \mu_0 \text{ and } \mu_1 \right\} \leq W_2^2(\mu_0, \mu_1).$$

The proof of the converse part is more difficult. We need some preparation. First, we recall an elementary result in ODE.

Proposition 2.3 *Let Z_t be a Borel vector field satisfying the condition*

$$\int_0^T \left(\sup_{x \in B} |Z_t(x)| + Lip(Z_t, B))dt \right) < +\infty \tag{2.33}$$

where $Lip(Z_t, B)$ denotes the local Lipschitz constant in the ball B. Then for $x \in \mathbb{R}^d$ and $s \in [0, T]$, the ODE

$$\frac{dX_t(x,s)}{dt} = Z_t(X_t(x,s)), \quad X_s(x,s) = x \tag{2.34}$$

admits a unique solution in an interval $I(x,s) \supset (s-\delta, s+\delta)$. Furthermore, if

$$\sup_{t \in I(x,s)} |X_t(x,s)| < +\infty$$

then $I(x,s) = [0, T]$. Finally, if Z satisfies the global condition

$$S := \int_0^1 \left(\| Z_t \|_{L^\infty} + Lip(Z_t, \mathbb{R}^d) \right) dt < +\infty, \tag{2.35}$$

then the flow X satisfies

$$\int_0^1 |\partial_t X_t(x,s)| dt \leq S, \sup_{s,t \in [0,1]} Lip(X_t(\cdot, s); \mathbb{R}^d)) \leq e^S. \tag{2.36}$$

Proof. Let's check the second term of (2.36). We have, for $x, y \in \mathbb{R}^d$,

$$|X_t(x,s) - X_t(y,s)| \le |x-y| + \int_s^t Lip(Z_\tau, \mathbb{R}^d)|X_\tau(x,s) - X_\tau(y,s)|d\tau.$$

The Gronwall lemma gives

$$|X_t(x,s) - X_t(y,s)| \le |x-y| \cdot e^{\int_s^t Lip(Z_\tau,\mathbb{R}^d)d\tau} \le |x-y|e^S.$$

□

Note: This proposition deals with the case where $t \mapsto Z_t$ is not continuous. If Z_t satisfies the global condition (2.36), then for any $t \in I(x,s)$,

$$\begin{aligned}|X_t(x,s)| &\le |x| + \int_s^t |Z_\tau(X_\tau(x,s))|d\tau \\ &\le |x| + \int_s^t |||Z_\tau||_{L^\infty} d\tau \le |x| + S < +\infty;\end{aligned}$$

therefore the life time $\tau_{x,s} = +\infty$ on $I(x,s) = [0,T]$.

Proposition 2.4 *Let $\psi \in C_b^1((0,1) \times \mathbb{R}^d)$ and $f \in C_b^1(\mathbb{R}^d)$. Then there exists a solution u_t to*

$$\partial_t u + \langle Z_t, \nabla u_t\rangle = \psi \quad on \quad (0,1)\times\mathbb{R}^d \tag{2.37}$$

with $u_t|_{t=1} = f$.

Proof. For $0 < t < 1$, set

$$\varphi(t,x) = f(X_1(x,t)) - \int_t^1 \psi(s, X_s(x,t))ds.$$

Note that $t \mapsto \varphi(t,x) \notin C^1(\mathbb{R}^d)$, but absolutely continuous and $x \mapsto \varphi(t,x)$ is Lipschitz. Since $X_s(x,t)$ enjoys the flow property:

$$X_t(X_s(x,0), s) = X_t(x,0), \quad 0 < s < t,$$

then

$$\varphi(t, X_t(x,0)) = f(X_1(x,0)) - \int_t^1 \psi(s, X_s(x,0)ds$$

Taking the derivative with respect to t in the two sides, we get

$$(\partial_t\varphi + \langle\nabla\varphi, Z_t\rangle)(t, X_t(x,0)) = \psi(t, X_t(x,0))$$

but for $t \in (0,1)$ given, $x \mapsto X_t(x,0)$ is a global homeomorphism of $\mathbb{R}^d$, therefore φ is a solution to (2.37).

Under the condition (2.33) and assume that $\tau_x \in [0,T]$ for all $x \in \mathbb{R}^d$. Then for any $\mu_0 \in \mathbb{P}(\mathbb{R}^d)$, $\mu_t = (X_t)_*\mu_0$ satisfy the continuity equation $\frac{d\mu_t}{dt} + \nabla\cdot(Z_t\mu_t) = 0$.

In fact, for $\varphi \in C_c(\mathbb{R}^d)$, $t \mapsto \varphi(X_t)$ is absolutely continuous since for a.e. t,

$$\frac{d}{dt}\varphi(X_t(x)) = \langle\nabla\varphi(X_t(x)), Z_t(X_t(x))\rangle$$

and

$$\int_0^1 |\langle \nabla\varphi(X_t(x)), Z_t(X_t(x))\rangle|\, dt \leq \| \nabla\varphi \|_\infty \cdot \int_0^T \sup_B |Z_t|\, dt$$

where $B = supp(\varphi)$. Therefore

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \varphi d\mu_t &= \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(X_t(x)) d\mu_0(x) \\ &= \int_{\mathbb{R}^d} \langle \nabla\varphi(X_t(x)), Z_t(X_t(x))\rangle d\mu_0(x) = \int_{\mathbb{R}^d} \langle \nabla\varphi, Z_t\rangle d\mu_t\end{aligned}$$

which implies that μ_t satisfies the continuity equation. □

Theorem 2.3 *(Representation formula for the continuity equation). Let $t \mapsto \mu_t \in \mathbb{P}(\mathbb{R}^d)$ be weakly continuous. Suppose that*

$$\int_0^1 (\sup_B |Z_t| + Lip(Z_t, B))dt < \infty \quad \text{and} \quad \int_0^1 \int_{\mathbb{R}^d} |Z_t| d\mu_t dt < +\infty, \tag{2.38}$$

and

$$\frac{d\mu_t}{dt} + \nabla \cdot (Z_t \mu_t) = 0 \quad \text{on } (0,1) \times \mathbb{R}^d. \tag{2.39}$$

Then for μ_0-a.s. $x \in \mathbb{R}^d$, $X_t(x, 0)$ does not explode for $t \in [0,1]$ and $\mu_t = (X_t)_* \mu_0$.

Proof. See Ambrosio, Gigli and Savaré's book [1], Proposition 8.18 p.175. □

Proof of Theorem 2.2 First we regularize (μ_t) and (Z_t). Consider the Gauss kernel

$$\rho_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}$$

and set

$$\mu_t^\varepsilon = \mu_t * \rho_\varepsilon, \; E_t^\varepsilon = (Z_t \mu_t) * \rho_\varepsilon, \; Z_t^\varepsilon = \frac{E_t^\varepsilon}{\mu_t^\varepsilon}$$

where

$$\mu_t^\varepsilon = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) d\mu_t(y) \in C_b^\infty(\mathbb{R}^d)$$

$$E_t^\varepsilon = \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) Z_t(y) d\mu_t(y) \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$$

By the continuity of $(t,x) \mapsto \mu_t^\varepsilon(x)$ (which is left to the reader as an exercise),

$$\inf_{|x| \in \mathbb{R}, t \in [0,1]} \mu_t^\varepsilon(x) > 0.$$

Therefore Z_t^ε satisfies the first condition in (2.38). By the following Lemma 2.1

$$\int_0^1 \int_{\mathbb{R}^d} |Z_t^\varepsilon|^2 d\mu_t^\varepsilon dt \leq \int_0^1 \int_{\mathbb{R}^d} |Z_t|^2 d\mu_t dt < +\infty. \tag{2.40}$$

To apply Theorem 2.3, it is sufficient to check

$$\frac{d\mu_t^\varepsilon}{dt} + \nabla \cdot (Z_t^\varepsilon \mu_t^\varepsilon) = 0$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \langle \nabla\varphi, Z_t^\varepsilon \rangle d\mu_t^\varepsilon &= \int_{\mathbb{R}^d} \langle \nabla\varphi, E_t^\varepsilon \rangle dx \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla\varphi(x), Z_t(y)\rangle \rho_\varepsilon(x-y) d\mu_t(y) dx\end{aligned}$$

Doing the change of variable, $z = x - y$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla\varphi(x)\rho_\varepsilon(x-y)dx &= \int_{\mathbb{R}^d} \nabla\varphi(y+z)\rho_\varepsilon(z)dz \\ &= \int_{\mathbb{R}^d} \nabla\varphi(y-z)\rho_\varepsilon(z)dz = \nabla(\varphi * \rho_\varepsilon)(y)\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla\varphi(x), Z_t(y)\rangle \rho_\varepsilon(x-y) d\mu_t(y) dx = \int_{\mathbb{R}^d} \langle \nabla(\varphi * \rho_\varepsilon), Z_t\rangle d\mu_t(y)$$

Hence

$$\begin{aligned}&\int_0^1 \int_{\mathbb{R}^d} (-\alpha'(t)\varphi(x) + \alpha(t)\langle Z_t^\varepsilon, \nabla\varphi\rangle) d\mu_t^\varepsilon dt \\ =&\int_0^1 \int_{\mathbb{R}^d} (-\alpha'(t)\varphi * \rho_\varepsilon + \alpha(t)\langle \nabla(\varphi * \rho_\varepsilon), Z_t\rangle) d\mu_t(y) dt = 0,\end{aligned}$$

since $\int \varphi(x)d\mu_t^\varepsilon(x) = \int (\varphi * \rho_\varepsilon)(y)d\mu_t(y)$ and $\varphi * \rho_\varepsilon \in C_0^\infty(\mathbb{R}^d)$.
By representation Theorem 2.3, there exists a flow of measurable maps $X_t^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mu_t^\varepsilon = (X_t^\varepsilon)_* \mu_0^\varepsilon.$$

Define $\eta^\varepsilon \in \mathscr{C}(\mu_0^\varepsilon, \mu_1^\varepsilon)$ by

$$\int \psi(x,y) d\eta^\varepsilon(x,y) = \int_{\mathbb{R}^d} \psi(x, X_1^\varepsilon(x)) d\mu_0^\varepsilon(x).$$

Then

$$\begin{aligned}W_2^2(\mu_0^\varepsilon, \mu_1^\varepsilon) &\le \int_{\mathbb{R}^d} |X_1^\varepsilon(x) - x|^2 d\mu_0^\varepsilon(x) \\ &= \int_{\mathbb{R}^d} \left| \int_0^1 Z_s^\varepsilon(X_s^\varepsilon(x)) \right|^2 d\mu_0^\varepsilon(x) \\ &\le \int_0^1 \int_{\mathbb{R}^d} |Z_s^\varepsilon(X_s^\varepsilon(x))|^2 d\mu_0^\varepsilon(x) ds \\ &= \int_0^1 \int_{\mathbb{R}^d} |Z_s^\varepsilon|^2 d\mu_s^\varepsilon(x) ds \le \int_0^1 \int_{\mathbb{R}^d} |Z_t|^2 d\mu_t dt\end{aligned}$$

where the last inequality is deduced by (2.40).
The last part is to check that μ_t^ε converges to μ_t weakly: for $\varphi \in C_b(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi d\mu_t^\varepsilon &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x)\rho_\varepsilon(x-y)d\mu_t(y)dx \\ &= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\varphi(x)\rho_\varepsilon(x-y)dx\right]d\mu_t(y) \to \int_{\mathbb{R}^d}\varphi(y)d\mu_t(y) \quad \varepsilon\to 0\end{aligned}$$

Now letting $\varepsilon \downarrow 0$ in

$$W_2^2(\mu_0^\varepsilon,\mu_1^\varepsilon) \le \int_0^1\int_{\mathbb{R}^d}|Z_t|^2 d\mu_t dt,$$

we get

$$W_2^2(\mu_0,\mu_1) \le \int_0^1\int_{\mathbb{R}^d}|Z_t|^2 d\mu_t dt.$$

in fact, $(\mu,\nu)\mapsto W_2^2(\mu,\nu)$ is semi-lower continuous. □

Lemma 2.1 *We have*

$$\int_0^1\int_{\mathbb{R}^d}|Z_t^\varepsilon|^2 d\mu_t^\varepsilon dt \le \int_0^1\int_{\mathbb{R}^d}|Z_t|^2 d\mu_t dt < +\infty.$$

Proof.

$$Z_t^\varepsilon(x) = \int_{\mathbb{R}^d} Z_t(y)\frac{\rho_\varepsilon(x-y)d\mu_t(y)}{\mu_t^\varepsilon(x)}$$

By Jensen inequality

$$|Z_t^\varepsilon(x)|^2 \le \int_{\mathbb{R}^d}|Z_t(y)|^2\frac{\rho_\varepsilon(x-y)d\mu_t(y)}{\mu_t^\varepsilon(x)}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d}|Z_t^\varepsilon(x)|^2\mu_t^\varepsilon(x)dx &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}|Z_t^\varepsilon(y)|^2\rho_\varepsilon(x-y)d\mu_t(y)dx \\ &= \int_{\mathbb{R}^d}|Z_t(y)|^2 d\mu_t(y)\int_{\mathbb{R}^d}\rho_\varepsilon(x-y)dx = \int_{\mathbb{R}^d}|Z_t(y)|^2 d\mu_t(y)\end{aligned}$$

Integrating with respect to t, we get the result. □

For further development, we need the following result due to Brenier and McCann.

Theorem 2.4 *(Monge optimal map) Let μ_1, $\mu_2 \in \mathbb{P}_2(\mathbb{R}^d)$ such that the density with respect to the Lebesgue measure λ_d exists. Then there exists a unique invertible measurable map $I+T:\mathbb{R}^d\to\mathbb{R}^d$ such that*

$$\mu_2 = (I+T)_*\mu_1 \quad \text{and} \quad W_2^2(\mu_1,\mu_2) = \int_{\mathbb{R}^d}|T(x)|^2 d\mu_1(x)$$

As a byproduct of the proof of Theorem 2.4, in this case,

$$\mathscr{C}_0(\mu_1,\mu_2) = \{(I \times (I+T))_*\mu_1\}$$

In what follows, we will denote

$$\mathbb{P}_2^a(\mathbb{R}^d) = \{\mu \in \mathbb{P}_2(\mathbb{R}^d) : \frac{d\mu}{d\lambda_d} \ \text{ exists}\}$$

Proposition 2.5 *Let $\mu_1, \mu_2 \in \mathbb{P}_2^a(\mathbb{R}^d)$ and T given in Theorem 2.4. Then*

$$w_t := T(\tau_t^{-1}) \in T_{\nu_t} \ \ \textit{for} \ \ a.s. t \in (0,1)$$

where

$$\tau_t = I + tT \ \text{ and } \ \nu_t = (\tau_t)_*\mu_1.$$

Proof. We have

$$W_2^2(\mu_1,\nu_t) \le \int_{\mathbb{R}^d} |x-\tau_t(x)|^2 d\mu_1 = t^2 \int_{\mathbb{R}^d} |Tx|^2 d\mu_1(x)$$

or

$$W_2(\mu_1,\nu_t) \le tW_2(\mu_1,\mu_2).$$

$$\begin{aligned} W_2^2(\mu_2,\nu_t) &\le \int_{\mathbb{R}^d} |x-\tau_t\circ(T+I)^{-1}|^2 d\mu_2(x) \\ &= \int_{\mathbb{R}^d} |x+T(x)-\tau_t(x)|^2 d\mu_1(x) \\ &= (1-t)^2 \int_{\mathbb{R}^d} |T(x)|^2 d\mu_1(x) \end{aligned}$$

or

$$W_2(\mu_t,\nu_t) \le (1-t)W_2(\mu_1,\mu_2).$$

Therefore

$$W_2(\mu_1,\nu_t) = tW_2(\mu_1,\mu_2)$$

and τ_t is the Monge optimal map. By convexity of the entropy functional (see the next section), $\nu_t \in \mathbb{P}_2^a(\mathbb{R}^d)$ and τ_t^{-1} exists. Now for $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \varphi d\nu_t &= \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x+tT(x))d\mu_1(x) \\ &= \int_{\mathbb{R}^d} \langle \nabla\varphi(x+tT(x)), T(x)\rangle d\mu_1(x) \\ &= \int_{\mathbb{R}^d} \langle \nabla\varphi, T(\tau_t^{-1}(x))\rangle d\nu_t \end{aligned}$$

Let $Z_t = \frac{d^o\nu_t}{dt}$, then there exists a full measure set $\Omega_\varphi \subset (0,1)$ such that

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, W_t - Z_t\rangle d\nu_t = 0$$

Using the separability of $C_c^\infty(\mathbb{R}^d)$, there exist a full measure set $\Omega \subset (0,1)$ such that

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, W_t - Z_t\rangle d\nu_t = 0, \forall \varphi \in C_c^\infty(\mathbb{R}^d).$$

Then $\exists \eta_t \in L^2(\mathbb{R}^d, \mathbb{R}^d, \nu_t)$ orthogonal to T_{ν_t} such that

$$W_t = \eta_t + Z_t.$$

But

$$\int_{\mathbb{R}^d} |T(x)|^2 d\mu_1 = \int_{\mathbb{R}^d} |W_t|^2 d\nu_t = \int_{\mathbb{R}^d} |Z_t|^2 d\nu_t + \int_{\mathbb{R}^d} |\eta_t|^2 d\nu_t$$

$$\begin{aligned} &\Rightarrow \quad W_2^2(\mu_1,\mu_2) = \int_0^1 \int_{\mathbb{R}^d} |Z_t|^2 d\nu_t dt + \int_0^1 \int_{\mathbb{R}^d} |\eta_t|^2 d\nu_t dt \\ &\Rightarrow \quad \eta = 0 \end{aligned}$$

. □

3 Convex functionals on $\mathbb{P}_2(\mathbb{R}^d)$

The notion of convex functionals in Wasserstein spaces was first studied by McCann: They have deep applications in Functional inequalities, in gradient flows and in non-linear PDE.

Definition 3.1 (λ convexity along geodesics) *Let $\Phi : \mathbb{P}_2(\mathbb{R}^d) \mapsto (-\infty, \infty]$ be a semi-lower continuous functional and $\lambda \in \mathbb{R}$ be given. We say that Φ is λ-convex along geodesics if for any $\mu_1, \mu_2 \in Dom(\Phi)$, $\exists\ \gamma \in \mathscr{C}_0(\mu_1,\mu_2)$ such that*

$$\Phi(\mu_t^{1\to 2}) \le (1-t)\Phi(\mu_1) + t\Phi(\mu_2) - \frac{\lambda}{2} t(1-t) W_2^2(\mu_1,\mu_2), \tag{3.41}$$

where

$$\mu_t^{1\to 2} = ((1-t)\pi_1 + t\pi_2)_*\gamma. \tag{3.42}$$

In what follows, we will give an interesting example of geodesically convex functionals.

Example 3.1 *Let $F : [0,\infty) \to (-\infty,\infty]$ be a proper, lower semi-continuous convex function such that $F(0) = 0, \liminf_{s\downarrow 0} \frac{F(s)}{s^\alpha} > \infty$ for some $\alpha > \frac{d}{d+2}$. For example, (i) $F(s) = s\log s$, (ii) $F(s) = \frac{s^m}{m-1}$, $m > 1$ satisfy the above conditions. For such a function F, we define the functional $\mathcal{F} : \mathbb{P}_2(\mathbb{R}^d) \to (-\infty,\infty]$ by*

$$\mathcal{F}(\mu) = \begin{cases} \int_{\mathbb{R}^d} F(\rho(x)) d\lambda_d(x) & if \quad \rho = \frac{d\mu}{d\lambda_d} \\ \infty & otherwise. \end{cases}$$

Proposition 3.1 *If the map $s \mapsto s^d F(s^{-d})$ is convex and decreasing in $(0,\infty)$, then the functional $\mathcal{F}$ is convex along geodesics: $\forall \mu_1,\mu_2 \in \mathbb{P}_2(\mathbb{R}^d), \exists \gamma \in \mathscr{C}_0(\mu_1,\mu_2)$ such that*

$$\mathcal{F}(\mu_t^{1\to 2}) \le (1-t)\mathcal{F}(\mu_1) + t\mathcal{F}(\mu_2),$$

where $\mu_t^{1\to 2} = ((1-t)\pi_1 + t\pi_2)_*\gamma$.

Proof. The proof of this result uses sophisticated properties of Monge optimal transport maps, we refer the reader to [1], p.212. □

Remark 3.1 *For* $F(s) = s\log s,\ s^d F(s^{-d}) = -d\log s$ *is convex and decreasing.*
For $F(s) = \frac{s^m}{m-1},\ s^d F(s^{-d}) = \frac{s^{(1-m)d}}{m-1}$ *it is the same as above.*

Remark 3.2 *The two examples given above are among the most important in* $\mathbb{P}_2(\mathbb{R}^d)$ *: the gradient flow associated to* $s \mapsto s\log s$ *corresponds to the heat equation, while to* $\frac{s^m}{m-1}$ *the Porous medium equation.*

Remark on the convexity of $\mu \mapsto \frac{1}{2}W_2^2(\mu,\mu_0)$

Let's begin with the function $x \mapsto \frac{1}{2}x^2$ on $\mathbb{R}$. We have

$$((1-t)x+ty)^2 = (1-t)x^2 + ty^2 - t(1-t)(x-y)^2$$

or

$$\frac{1}{2}((1-t)x+ty)^2 = \frac{1}{2}(1-t)x^2 + \frac{1}{2}ty^2 - \frac{1}{2}t(1-t)(x-y)^2,$$

which is finer than the convex property of $x \mapsto \frac{1}{2}x^2$ In higher dimension, $\mathbb{R}^d$, the Hessian of $x \mapsto \frac{1}{2}|x|^2$ is Id, so we have that

$$\frac{1}{2}|(1-t)x+ty|^2 \le \frac{1}{2}(1-t)|x|^2 + \frac{1}{2}t|y|^2 - \frac{1}{2}t(1-t)|x-y|^2.$$

However for the Wasserstein distance, it has been noticed that $\mu \mapsto \frac{1}{2}W_2^2(\mu,\mu_0)$ is not 1-convex along geodesics(see [1], p.204), but 1-convex along an interpolating curve belonging to a larger class of curves: generalized geodesics.

Definition 3.2 *A generalized geodesic joining* μ_2 *to* μ_3 *(with base* μ_1*) is a curve*

$$\mu_t^{2\to3} := (\pi_t^{2\to3}) * \lambda$$

where $\lambda \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ *such that* $(\pi_1,\pi_2)_*\lambda \in \mathscr{C}_0(\mu_1,\mu_2)$, $(\pi_1,\pi_3)_*\lambda \in \mathscr{C}_0(\mu_1,\mu_3)$ *and* $\pi_t^{2\to3} = (1-t)\pi_2 + t\pi_3$.

Note that {geodesics}⊂ {generalized geodesics}. In fact, take $\mu_1 = \mu_2$ and $\gamma \in \mathscr{C}_0(\mu_2,\mu_3)$ and $\gamma_{11} \in \mathscr{C}_0(\mu_2,\mu_2)$. Then for $\hat{\mu} \in \Gamma(\mu_1,\mu_2,\mu_2)$ such that $(\pi_1,\pi_2)_*\hat{\mu} = \gamma_{11}$ and $(\pi_2,\pi_3)_*\hat{\mu} = \gamma$, we have

$$(\pi_t^{2\to3})_*\hat{\mu} = (\pi_t^{2\to3})_*\gamma.$$

□.

Convexity along generalized geodesics

We say that $\Phi : \mathbb{P}_2(\mathbb{R}^d) \mapsto (-\infty,\infty]$ is λ-convex along generalized geodesics if for any $\mu_1,\mu_2,\mu_3 \in Dom(\Phi)$, there exists a generalized geodesic $\mu_t^{2\to3}$ connecting μ_2 and μ_3 such that for all $t \in [0,1]$

$$\Phi(\mu_t^{2\to3}) \le (1-t)\Phi(\mu_2) + t\Phi(\mu_3) - \frac{\lambda t(1-t)}{2}W_2^2(\mu_2,\mu_3). \tag{3.43}$$

If $\lambda > 0$, a direct result of (3.43) is the uniqueness of the minimum of Φ over any "generalized convex" subset $C \subset Dom(\Phi)$.

Proposition 3.2 *We have that*

$$W_2^2(\mu_1, \mu_t^{2\to3}) \leq (1-t)W_2^2(\mu_1, \mu_2) + tW_2^2(\mu_1, \mu_3) - t(1-t)W_2^2(\mu_2, \mu_3).$$

Proof. Define $\mu_t^{1,2\to3} = ((1-t)\pi_{12} + t\pi_{13})_*\hat{\mu} \in \mathcal{C}(\mu_1, \mu_t^{2\to3})$, where $\pi_{12} = (\pi_1, \pi_2), \pi_{13} = (\pi_1, \pi_3)$. Then

$$\begin{aligned}
W_2^2(\mu_1, \mu_t^{2\to3}) &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |y_1 - y_2|^2 d\mu_t^{1,2\to3} \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} |(1-t)(x_1-x_2) + t(x_1-x_3)|^2 d\hat{\mu}(x_1,x_2,x_3) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} ((1-t)|x_2-x_1|^2 + t|x_3-x_1|^2 - t(1-t)|x_2-x_3|^2) d\hat{\mu}(x_1,x_2,x_3) \\
&\leq (1-t)W_2^2(\mu_1,\mu_2) + tW_2^2(\mu_1,\mu_3) - t(1-t)W_2^2(\mu_2,\mu_3).
\end{aligned}$$

The result follows. □

Entropy functionals and log-concave measures

Let γ, μ be Borel probability measures on $\mathbb{R}^d$, the relative entropy of μ with respect to γ is defined by

$$Ent_\gamma(\mu) = \begin{cases} \int_{\mathbb{R}^d} \rho \log \rho d\gamma & if \quad d\mu = \rho d\gamma \\ \infty & otherwise. \end{cases}$$

Introduce the function

$$H(s) = \begin{cases} s(\log s - 1) + 1 & if \quad s \geq 0, \\ \infty & if \quad s < 0. \end{cases}$$

$s \mapsto H(s)$ is lower semi-continuous, strictly convex function on $\mathbb{R} \to [0, \infty]$. Note that

$$Ent_\gamma(\mu) = \int_{\mathbb{R}^d} H(\rho(x))d\gamma \geq 0 \text{ and } Ent_\gamma(\mu) = 0 \Leftrightarrow \rho(x) \equiv 1.$$

Now we consider $\gamma = Ce^{-V}\lambda_d \in \mathbb{P}(\mathbb{R}^d)$.

Proposition 3.3

$$Ent_\gamma(\mu) = \mathcal{F}(\mu) + \int_{\mathbb{R}^d} V(x)d\mu(x) - \log C,$$

where

$$\mathcal{F}(\mu) = \begin{cases} \int_{\mathbb{R}^d} \rho(x)\log\rho(x)d\lambda_d(x) & if \quad \mu = \rho\lambda_d, \\ \infty & otherwise. \end{cases}$$

Proof. let $\mu = \rho\gamma = \rho Ce^{-V}\lambda_d$. We have that

$$\begin{aligned}
\mathcal{F}(\mu) &= \int_{\mathbb{R}^d} \rho Ce^{-V} \log(\rho Ce^{-V}) d\lambda_d \\
&= \int_{\mathbb{R}^d} \rho \log \rho d\gamma + \int_{\mathbb{R}^d} \log(Ce^{-V}) d\mu \\
&= Ent_\gamma(\mu) - \int_{\mathbb{R}^d} V(x)d\mu(x) + \log C.
\end{aligned}$$

□

Proposition 3.4 *Suppose $V(x) \geq -A - B|x|^2$ and for $x, y \in \mathbb{R}^d$,*

$$V((1-t)x + ty) \leq (1-t)V(x) + tV(y) - \frac{\lambda t(1-t)}{2}|x-y|^2,$$

then the functional

$$\mu \mapsto \mathcal{F}_2(\mu) := \int_{\mathbb{R}^d} V(x) d\mu(x)$$

is λ-convex along all geodesics; along all generalized geodesics if $\lambda \geq 0$.

Proof. Note that for $\mu \in \mathbb{P}_2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} V(x) d\mu(x) \geq -A - Bm_2(\mu) > -\infty.$$

So the functional $\mathcal{F}_2 : \mathbb{P}_2(\mathbb{R}^d) \to (-\infty, \infty]$. Now let $\mu_1, \mu_2 \in \mathbb{P}_2(\mathbb{R}^d)$ and $\gamma \in \mathscr{C}_0(\mu_1, \mu_2)$. Consider the geodesic

$$\mu_t = ((1-t)\pi_1 + t\pi_2)_* \gamma.$$

Then

$$\mathcal{F}_2(\mu_t) = \int_{\mathbb{R}^d} V(x) d\mu_t(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} V((1-t)x + ty) d\gamma(x, y)$$

which is smaller, by λ-convexity of V, than

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}^d} ((1-t)V(x) + tV(y) - \frac{\lambda t(1-t)}{2}|x-y|^2) d\gamma(x, y) \\
&\quad = (1-t) \int_{\mathbb{R}^d} V(x) d\mu_1(x) + t \int_{\mathbb{R}^d} V(x) d\mu_2(x) - \frac{\lambda t(1-t)}{2} \int_{\mathbb{R}^d} |x-y|^2 d\gamma(x, y) \\
&\quad = (1-t)\mathcal{F}_2(\mu_1) + t\mathcal{F}_2(\mu_2) - \frac{\lambda t(1-t)}{2} W_2^2(\mu_1, \mu_2).
\end{aligned}$$

We prove the λ-convexity along geodesics. Let's see the λ-convexity along generalized geodesics. Let $\mu_0 \in \mathbb{P}_2(\mathbb{R}^d)$ be arbitrary, consider $\Gamma \in \Gamma(\mu_0, \mu_1, \mu_2)$ such that

$$(\pi_1, \pi_2)_* \Gamma \in \mathscr{C}_0(\mu_0, \mu_1), \ (\pi_1, \pi_3)_* \Gamma \in \mathscr{C}_0(\mu_1, \mu_2).$$

Let $\mu_t^{1 \to 2} = ((1-t)\pi_2 + t\pi_3)_* \Gamma$. We have that

$$\begin{aligned}
\mathcal{F}_2(\mu_t^{1 \to 2}) &= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} V((1-t)y + tz) d\Gamma(x, y, z) \\
&\leq (1-t) \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} V(y) d\Gamma(x, y, z) + t \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} V(z) d\Gamma(x, y, z) \\
&\quad - \frac{\lambda t(1-t)}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |y-z|^2 d\Gamma(x, y, z) \\
&\leq (1-t)\mathcal{F}_2(\mu_1) + t\mathcal{F}_2(\mu_2) - \frac{\lambda t(1-t)}{2} W_2^2(\mu_1, \mu_2),
\end{aligned}$$

since $(\pi_2, \pi_3)_* \Gamma \in \mathscr{C}(\mu_1, \mu_2)$. □

Corollary 3.1 *Let* $\gamma = \frac{e^{-\frac{|x|^2}{2}}}{(\sqrt{2\pi})^d}\lambda_d$ *be the standard Gaussian measure. Then* $\mu \mapsto Ent_\gamma(\mu)$ *is 1-convex along all generalized geodesics.*

Proof. The proof consists of two parts, the easy part concerns the functional $\mathcal{F}_2$, where $V(x) = -\frac{|x|^2}{2}$, which is 1-convex; the difficult part concerns $\mathcal{F}$ with $F(s) = s\log s$, which is, by Proposition 3.1, convex along all generalized geodesics. □

Gradient flows associated to a convex functional on $\mathbb{R}^d$

In the remain part of this section, we would like to emphasize the important role of convex functionals. Let's discuss only the case of $\mathbb{R}^d$. First, let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be C^2 such that

$$Hess(\Phi) = \left(\frac{\partial^2\Phi}{\partial x_i \partial x_j}\right) \geq \lambda Id,\ \lambda > 0, \tag{3.44}$$

then

$$\Phi((1-t)x + ty) \leq (1-t)\Phi(x) + t\Phi(y) - \frac{\lambda t(1-t)}{2}|x-y|^2. \tag{3.45}$$

Consider the differential equation

$$\frac{dX_t}{dt} = -(\nabla\Phi)(X_t),\ X_t|_{t=0} = x.$$

Then we have

$$\frac{d}{dt}\Phi(X_t) =< \nabla\Phi(X_t), \frac{dX_t}{dt} >= -|\nabla\Phi(X_t)|^2 \leq 0;$$

Therefore

$$\Phi(X_t) \leq \Phi(x) \text{ for all } t \geq 0$$

implying that X_t does not explode. Now we compute

$$\frac{d}{dt}|X_t(x) - X_t(y)|^2 = -2 < X_t(x) - X_t(y), \nabla\Phi(X_t(x)) - \nabla\Phi(X_t(y)) > \tag{3.46}$$

but

$$\nabla\Phi(X_t(x)) - \nabla\Phi(X_t(y)) = \left(\int_0^1 Hess\Phi((1-s)X_t(y) + sX_t(x))ds\right)(X_t(x) - X_t(y)).$$

Combining (3.44) with (3.46), we get

$$\frac{d}{dt}|X_t(x) - X_t(y)|^2 \leq -2\lambda|X_t(x) - X_t(y)|^2 \tag{3.47}$$

which implies that

$$|X_t(x) - X_t(y)|^2 \leq e^{-2\lambda t}|x-y|^2$$

or

$$|X_t(x) - X_t(y)| \leq e^{-\lambda t}|x-y|. \tag{3.48}$$

Now for a general convex functional Φ satisfying (3.45), the gradient is replaced by the notion of sub-gradient: we say that $v \in \mathbb{R}^d$ is a sub-gradient of Φ at x if $\Phi(x+y) \geq \Phi(x) + < v, y > +o(|y|)$, as $y \to 0$. We denote by $\partial\Phi(x) = \{\text{subgradients of } \Phi \text{ at } x\}$ which is a convex subset of $\mathbb{R}^d$.
A result in convex analysis says that for a lower semi-continuous convex function Φ, $\nabla\Phi(x)$ exists for a.e. $x \in \mathbb{R}^d$ and $\partial\Phi(x) \neq \emptyset$ for each $x \in \mathbb{R}^d$.

Definition 3.3 *We say that $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is a gradient flow associated to Φ if $t \mapsto X_t$ is absolutely continuous and*

$$\frac{dX_t(x)}{dt} \in \partial\Phi(X_t(x)).$$

Theorem 3.1 (De Giorgi) *If Φ is λ-convex with $\lambda \geq 0$, then*

$$|X_t(x) - X_t(y)| \leq e^{-\lambda t}|x - y|.$$

4 Gradient flow associated to the entropy functionals

The general theory of gradient flows associated to convex functionals on $\mathbb{P}_2(\mathbb{R}^d)$ is well established in [1], and also complicated. To simplify the things, we take the entropy functional

$$\mu \mapsto Ent_{\gamma_d}(\mu)$$

where γ_d =*standard Gaussian measure on* $\mathbb{R}^d$. By the discussion in Section 3, it is 1-convex along all generalized geodesics. In what follows, we denote

$$P^*(\mathbb{R}^d) = \{\mu \in \mathbb{P}_2(\mathbb{R}^d) : Ent_{\gamma_d}(\mu) < \infty\}.$$

Then $Ent_{\gamma_d} : P^*(\mathbb{R}^d) \to [0, \infty)$.

Proposition 4.1 *Let Z be a smooth vector field on $\mathbb{R}^d$ with compact support and $(U_t)_{t\in R}$ be the flow of diffeomorphisms associated to Z:*

$$\frac{dU_t(x)}{dt} = Z(U_t(x)), \quad U_0(x) = x$$

Then

$$(U_t)_*\gamma_d = K_t \cdot \gamma_d,$$

with

$$K_t(x) = \exp\left(\int_0^t \mathrm{div}_{\gamma_d}(Z)(U_{-s}(x))ds\right)$$

where $\mathrm{div}_{\gamma_d}(Z)$ is the divergence of Z, relative to γ_d:

$$\int_{\mathbb{R}^d} \langle\nabla\varphi, Z\rangle d\gamma_d = \int_{\mathbb{R}^d} \varphi\, \mathrm{div}_{\gamma_d}(Z) d\gamma_d, \quad \varphi \in C_b^1(\mathbb{R}^d),$$

we have

$$\mathrm{div}_{\gamma_d}(Z) = \sum_{i=1}^d \left(x_i Z^i(x) - \frac{\partial Z^i(x)}{\partial x_i}\right).$$

Proof. Let $\varphi \in C_b^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \varphi(x) K_t(x) d\gamma_d(x) = \int_{\mathbb{R}^d} \varphi(U_t(x)) d\gamma_d(x)$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d}\varphi(x)K_t(x)d\gamma_d(x)\right)\bigg|_{t=0} = \int_{\mathbb{R}^d}\langle\nabla\varphi, Z\rangle d\gamma_d$$
$$= \int_{\mathbb{R}^d}\varphi\,\mathrm{div}_{\gamma_d}(Z)d\gamma_d$$

which implies that

$$\frac{dK_t(x)}{dt}\bigg|_{t=0} = \mathrm{div}_{\gamma_d}(Z).$$

Now using the flow property $U_{t+s} = U_t \circ U_s$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(x)K_t(x)d\gamma_d(x) &= \frac{d}{d\varepsilon}\int_{\mathbb{R}^d}\varphi(U_t(U_\varepsilon))d\gamma_d\\ &= \int_{\mathbb{R}^d}\varphi(U_t)\mathrm{div}_{\gamma_d}(Z)d\gamma_d\\ &= \int_{\mathbb{R}^d}\varphi\cdot\mathrm{div}_{\gamma_d}(Z)(U_{-t})\cdot K_t d\gamma_d\end{aligned}$$

It follows that

$$\frac{dK_t}{dt} = \mathrm{div}_{\gamma_d}(Z)(U_{-t})K_t, \quad K_0 = 1$$

which implies that

$$K_t = \exp\left(\int_0^t \mathrm{div}_{\gamma_d}(Z)(U_{-s})ds\right).$$

□

Since $T_\mu = \overline{\{\nabla F : F \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu)}$, we will consider $Z = \nabla F$ and U_t the associated flow.

Proposition 4.2 *Let $\mu_0 \in \mathbb{P}^*(\mathbb{R}^d)$ be given and $\mu_t = (U_t)_*(\mu_0)$. Then*

$$\frac{d}{dt}Ent_{\gamma_d}(\mu_t)|_{t=0} = \int_{\mathbb{R}^d} LF d\mu_0$$

where $LF = div_{\gamma_d}(\nabla F)$ which admits the expression

$$LF = -\sum_{i=1}^d \frac{\partial^2 F}{\partial x_i^2} + \sum_{i=1}^d x_i\frac{\partial F}{\partial x_i}$$

Proof. Let $\mu_0 = \rho_0\gamma_d$, then for $\varphi \in C_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\varphi\mu_t = \int_{\mathbb{R}^d}\varphi(U_t)\rho_0 d\gamma_d = \int_{\mathbb{R}^d}\varphi\rho_0(U_{-t})K_t d\gamma_d$$

It follows that

$$\mu_t = \rho_0(U_{-t})K_t\cdot\gamma_d := \rho_t\cdot\gamma_d$$

Then

$$\begin{aligned}Ent_{\gamma_d}(\mu_t) &= \int_{\mathbb{R}^d}\rho_0(U_{-t})K_t\lg(\rho_0(U_{-t})K_t)d\gamma_d\\ &= \int_{\mathbb{R}^d}\rho_0\lg(\rho_0(U_{-t})K_t)d\gamma_d\\ &= Ent_{\gamma_d}(\mu_0) + \int_{\mathbb{R}^d}\lg K_t(U_t)\cdot\rho_0 d\gamma_d\end{aligned}$$

By the expression of K_t,

$$\lg K_t(U_t) = \int_0^t (LF)(U_{t-s}(x))ds$$

Formally

$$\frac{d}{dt} Ent_{\gamma_d}(\mu_t)\Big|_{t=0} = \int_{\mathbb{R}^d} LF d\mu_0.$$

To make the computation rigorous, we need the estimate:

$$\| K_t \|_{L^p}^p \leq \int_{\mathbb{R}^d} \exp\left(\frac{p^2 T}{p-1} |\mathrm{div}_{\gamma_d}(Z)|\right) d\gamma_d, \quad t \leq T. \tag{4.49}$$

By expression of LF, there exists a small $\varepsilon_0 > 0$ such that

$$\int_{\mathbb{R}^d} e^{2\varepsilon_0 |LF|^2} d\gamma_d < +\infty$$

Set $u_t = \int_0^t \frac{1}{t}(LF)(U_{t-s}(x))ds$, by Jensen inequality,

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{\varepsilon_0 |u_t|^2} &\leq \int_{\mathbb{R}^d} \left(\frac{1}{t}\int_0^t e^{\varepsilon_0 |LF(U_{t-s})|^2} ds\right)^2 d\gamma_d \\
&= \frac{1}{t}\int_0^t \left(\int_{\mathbb{R}^d} e^{\varepsilon_0 |LF|^2} \cdot K_{t-s} d\gamma_d\right) ds \\
&\leq \left(\int_{\mathbb{R}^d} e^{2\varepsilon_0 |LF|^2} d\gamma_d\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^d} e^{4|LF|} d\gamma_d\right)^{\frac{1}{2}}
\end{aligned}$$

according to (4.49) for $p = 2$ and K_{t-s}. Now by Young inequality

$$\begin{aligned}
\int_{\mathbb{R}^d} |u_t|^2 \rho_0 d\gamma_d &\leq \int_{\mathbb{R}^d} \left(e^{q_0 |u_t|^2} + \frac{\rho_0}{\varepsilon_0} \lg \frac{\rho_0}{\varepsilon_0}\right) d\gamma_d \\
&= \int_{\mathbb{R}^d} e^{\varepsilon_0 |u_t|^2} d\gamma_d + \frac{1}{\varepsilon_0} Ent_{\gamma_d}(\mu_0) - \frac{\lg \varepsilon_0}{\varepsilon_0}
\end{aligned}$$

Combining with the above estimate, we get

$$\sup_{0 \leq t \leq 1} \int_{\mathbb{R}^d} |u_t|^2 \rho_0 d\gamma_d < +\infty$$

Therefore we can take the limit under the integral, the proof is completed. □

We will denote by

$$(\partial_{\nabla F} Ent_{\gamma_d})(\mu_0) = \frac{d}{dt}|_{t=0} Ent_{\gamma_d}(\mu_t).$$

Example 4.1 *Let* $\rho_0 \geq \varepsilon_0$ *and* $\rho_0 \in C_b^\infty(\mathbb{R}^d)$. *Then* $\lg \rho_0$, $\nabla(\lg \rho_0) \in L^2(\mathbb{R}^d, \gamma_d)$.

We say that $\lg \rho_0 \in \mathbb{D}_1^2(\mathbb{R}^d, \gamma_d)$. Then there exists $\varphi_n \in C_c^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \left(|\varphi_n - \lg \rho_0|^2 + |\nabla \varphi_n - \nabla \lg \rho_0|^2\right) d\gamma_d \to 0.$$

In particular,

$$\int_{\mathbb{R}^d} |\nabla\varphi_n - \nabla \lg \rho_0|^2 \cdot \rho_0 d\gamma_d \leq \| \rho_0 \|_{L^\infty} \int_{\mathbb{R}^d} |\nabla\varphi_n - \nabla \lg \rho_0|^2 d\gamma_d \to 0.$$

therefore $\nabla \lg \rho_0 \in T_{\mu_0}$. Now

$$\begin{aligned}(\partial_{\nabla F} Ent_{\gamma_d})(\mu_0) &= \int_{\mathbb{R}^d} \mathrm{div}_{\gamma_d}(\nabla F)\rho_0 d\gamma_d \\ &= \int_{\mathbb{R}^d} \langle \nabla F, \nabla \rho_0\rangle d\gamma_d = \int_{\mathbb{R}^d} \langle \nabla F, \nabla \lg \rho_0\rangle d\mu_0.\end{aligned}$$

Definition 4.1 *We say that the gradient ∇Ent_{γ_d} exists at $\mu_0 \in \mathbb{P}^*(\mathbb{R}^d)$ if there exists $v \in T_{\mu_0}$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$(\partial_{\nabla\varphi} Ent_{\gamma_d})(\mu_0) = \langle v, \nabla\varphi\rangle_{T_{\mu_0}}.$$

It is clear that v is uniquely determined and we will denote

$$v = \nabla Ent_{\gamma_d}(\mu_0) \in T_{\mu_0}.$$

Theorem 4.1 *Let $\mu_0 \in \mathbb{P}^*(\mathbb{R}^d)$. Then for any $\eta > 0$, there exists a unique $\hat{\mu} \in \mathbb{P}^*(\mathbb{R}^d)$ such that*

$$\frac{1}{2}W_2^2(\mu_0, \hat{\mu}) + \eta Ent_{\gamma_d}(\hat{\mu}) = \inf\left\{\frac{1}{2}W_2^2(\mu_0, \mu) + \eta Ent_{\gamma_d}(\mu) : \mu \in \mathbb{P}^*(\mathbb{R}^d)\right\}$$

and the gradient ∇Ent_{γ_d} exists at $\hat{\mu}$

Proof. **Uniqueness** of $\hat{\mu}$. Suppose that there are two measures $\hat{\mu}_1$, $\hat{\mu}_2$ which realize the minimum. By Proposition 3.2, there exists a generalized geodesic $\hat{\mu}_t$ jointing $\hat{\mu}_1$, $\hat{\mu}_2$ such that

$$\frac{1}{2}W_2^2(\mu_0, \hat{\mu}_t) \leq (1-t)\frac{1}{2}W_2^2(\mu_0, \hat{\mu}_1) + \frac{1}{2}W_2^2(\mu_0, \hat{\mu}_2) - \frac{t(1-t)}{2}W_2^2(\hat{\mu}_1, \hat{\mu}_2)$$

By Corollary 3.1,

$$Ent_{\gamma_d}(\hat{\mu}_t) \leq (1-t)Ent_{\gamma_d}(\hat{\mu}_1) + tEnt_{\gamma_d}(\hat{\mu}_2) - \frac{t(1-t)}{2}W_2^2(\hat{\mu}_1, \hat{\mu}_2),$$

It follows that

$$\frac{1}{2}W_2^2(\mu_0, \hat{\mu}_t) + \eta Ent_{\gamma_d}(\hat{\mu}_t) < minimum$$

which yields the contradiction.
Existence Let

$$m = \inf\left\{\frac{1}{2}W_2^2(\mu_0, \mu) + \eta Ent_{\gamma_d}(\mu) : \mu \in P^*(\mathbb{R}^d)\right\}$$

which is finite. Then for $n \geq 1$, $\exists \mu_n \in \mathbb{P}^*(\mathbb{R}^d)$ such that

$$\frac{1}{2}W_2^2(\mu_0, \mu_n) + \eta Ent_{\gamma_d}(\mu_n) \leq m + \frac{1}{n} \leq m + 1 \tag{4.50}$$

From which we deduce that $\sup_n W_2^2(\mu_0, \mu_n) < +\infty$ so that

$$\sup_n \int_{\mathbb{R}^d} |x|^2 d\mu_n < +\infty.$$

Therefore the family $\{\mu_n : n \geq 1\}$ is tight. Up to a subsequence, μ_n converges to $\hat{\mu} \in \mathbb{P}_2(\mathbb{R}^d)$. We will prove that $\hat{\mu} \in \mathbb{P}^*(\mathbb{R}^d)$. Let

$$C = \sup_{n\geq 1} Ent_{\gamma_d}(\mu_n) < \infty.$$

Let $\mu_n = \rho_n \gamma_d$. We have

$$\int_{\rho_n \geq R} \rho_n \, d\gamma_d \leq \frac{1}{\log R} \int_{\rho_n \geq R} \rho_n \log \rho_n \, d\gamma_d.$$

But

$$\begin{aligned} Ent_{\gamma_d}(\rho) &= \int_{\mathbb{R}^d} \rho \log \rho \, d\gamma_d \\ &= \int_{0\leq\rho\leq 1} \rho \log \rho \, d\gamma_d + \int_{\{\rho\geq 1\}} \rho \log \rho \, d\gamma_d \\ &\geq -\frac{1}{e} + \int_{\{\rho\geq 1\}} \rho \log \rho \, d\gamma_d, \end{aligned}$$

since $min_{0\leq s\leq 1}(slogs) = -\frac{1}{e}$. Then for $R \geq 1$,

$$\int_{\{\rho\geq R\}} \rho \lg \rho d\gamma_d \leq \int_{\{\rho\geq 1\}} \rho \lg \rho d\gamma_d \leq Ent_{\gamma_d}(\rho) + \frac{1}{e}$$

Therefore

$$\sup_n \int_{\{\rho\geq R\}} \rho \lg \rho d\gamma_d \leq \frac{1}{\lg R}(C + \frac{1}{e}) \to 0 \ \text{ as } R \to \infty \tag{4.51}$$

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel function. Then there is a constant C_ψ such that for $\delta > 0$, $\exists \varphi \in C_b(\mathbb{R}^d)$, $\| \varphi \|_\infty \leq C_\psi$ and

$$\int_{\mathbb{R}^d} |\psi - \varphi| d\gamma_d < \delta, \quad \int_{\mathbb{R}^d} |\psi - \varphi| d\hat{\mu} < +\infty.$$

Hence

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \psi \rho_n d\gamma_d - \int_{\mathbb{R}^d} \psi d\hat{\mu} \right| \leq & \int_{\mathbb{R}^d} |\varphi - \psi| \rho_n d\gamma_d + \int_{\mathbb{R}^d} |\psi - \varphi| d\hat{\mu} \\ & + \left| \int_{\mathbb{R}^d} \varphi \rho_n d\gamma_d - \int_{\mathbb{R}^d} \varphi d\hat{\mu} \right| \end{aligned}$$

the first term in the right side,

$$\begin{aligned} \int_{\mathbb{R}^d} |\psi - \varphi| \rho_n d\gamma_d &\leq R \cdot \int_{\{\rho_n \leq R\}} |\varphi - \psi| d\gamma_d + \int_{\{\rho_n > R\}} |\varphi - \psi| \rho_n d\gamma_d \\ &\leq R \cdot \delta + 2C_\psi \cdot \int_{\{\rho_n > \mathbb{R}\}} \rho_n d\gamma_d \end{aligned}$$

Let $\varepsilon > 0$, By (4.50), take R big enough such that

$$2C_\psi \cdot \int_{\{\rho_n > \mathbb{R}\}} \rho_n d\gamma_d < \frac{\varepsilon}{4}$$

Choose $\delta < \frac{\varepsilon}{4R}$, then we get $\int_{\mathbb{R}^d} |\psi - \varphi| \rho_n d\gamma_d < \frac{\varepsilon}{2}$, for all n. Now for n big enough, the last term in (4.51) is smaller than $\frac{\varepsilon}{4}$, so we have for $n \geq n_0$ big enough,

$$\left| \int_{\mathbb{R}^d} \psi \rho_n d\gamma_d - \int_{\mathbb{R}^d} \psi d\hat{\mu} \right| < \varepsilon.$$

This means that for any bounded function ϕ,

$$\int_{\mathbb{R}^d} \psi d\hat{\mu} = \lim_{n\to\infty} \int_{\mathbb{R}^d} \psi \rho_n d\gamma_d.$$

In particular, for $E \in \mathcal{B}(\mathbb{R}^d)$, $\gamma_d(E) = 0$, we have $\hat{\mu}(E) = 0$. In other words, $d\hat{\mu} = \hat{\rho} \cdot d\gamma_d$. Now,

$$Ent_{\gamma_d}(\hat{\rho}) \leq \liminf_{n\to\infty} Ent_{\gamma_d}(\rho_n) \leq C < +\infty.$$

Now using again the semi-lower continuity of

$$\mu \mapsto \frac{1}{2} W_2^2(\mu_0, \mu) + \eta Ent_{\gamma_d}(\mu)$$

We get

$$\frac{1}{2} W_2^2(\mu_0, \mu) + \eta Ent_{\gamma_d}(\hat{\mu}) = m.$$

In the last part, we will prove that $(\nabla Ent_{\gamma_d})(\hat{\mu})$ exists. Let (U_t) be the flow associated to ∇F with $F \in C_c^\infty(\mathbb{R}^d)$. Let $\Gamma \in \mathscr{C}_0(\mu_0, \hat{\mu})$ and define $\Gamma_t \in \mathscr{C}(\mu_0, (U_t)_* \hat{\mu})$ by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, y) d\Gamma_t = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, U_t(y)) \Gamma(dx, dy).$$

We have

$$W_2^2(\mu_0, (U_t)_* \hat{\mu}) - W_2^2(\mu_0, \hat{\mu}) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x - U_t(y)|^2 - |x - y|^2 \right) \Gamma(dx, dy)$$

then

$$\lim_{t\to 0} \frac{1}{2t} \left[W_2^2(\mu_0, (U_t)_* \hat{\mu}) - W_2^2(\mu_0, \hat{\mu}) \right] \leq - \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle x - y, Z(y) \rangle \Gamma(dx, dy), \tag{4.52}$$

where $Z = \nabla F$. On the other hand, by construction of $\hat{\mu}$, for $t > 0$,

$$0 \leq \frac{\eta}{t} [Ent_{\gamma_d}((U_t)_* \hat{\mu}) - Ent_{\gamma_d}(\hat{\mu})] + \frac{1}{2t} \left[W_2^2(\mu_0, (U_t)_* \hat{\mu}) - W_2^2(\mu_0, \hat{\mu}) \right]$$

Letting $t \to 0$, the first term tends to $\eta \cdot (\partial_{\nabla F} Ent_{\gamma_d})(\hat{\mu})$. Combining with (4.52), we get

$$0 \leq \eta \cdot (\partial_{\nabla F} Ent_{\gamma_d})(\hat{\mu}) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle x - y, Z(y) \rangle \Gamma(dx, dy)$$

Using Proposition 4.2,

$$\partial_{-\nabla F} Ent_{\gamma_d} = -\partial_{\nabla F} Ent_{\gamma_d},$$

Changing F into $-F$, the above inequality gives

$$(\partial_{\nabla F} Ent_{\gamma_d})(\hat{\mu}) = \frac{1}{\eta}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle x-y, Z(y)\rangle \Gamma(dx,dy) \tag{4.53}$$

Now by Brenier's result, $\Gamma = (I + (I+\xi))_*\mu_0$. The right hand of (4.53) is written

$$-\frac{1}{\eta}\int_{\mathbb{R}^d} \langle \xi(x), Z(x+\xi(x))\rangle d\mu_0 = -\frac{1}{\eta}\int_{\mathbb{R}^d} \langle \xi\circ\tau^{-1}(x), \nabla F(x)\rangle d\hat{\mu}(x).$$

where $\tau = I + \xi$. Note that

$$\int_{\mathbb{R}^d} |\xi\circ\tau^{-1}|^2 d\hat{\mu} = \int_{\mathbb{R}^d} |\xi|^2 d\mu_0 = W_2^2(\mu_0,\hat{\mu}) < +\infty$$

therefore $(\nabla Ent_{\gamma_d})(\hat{\mu})$ exists, which is the orthogonal projection of $-\frac{\xi\circ\tau^{-1}}{\eta}$ on $T_{\hat{\mu}}$. □

We will denote by

$$Dom(\nabla Ent_{\gamma_d}) = \{\nu \in \mathbb{P}^*(\mathbb{R}^d) : \nabla Ent_{\gamma_d}(\nu) \in T_\nu \ \text{ exists}\}.$$

Now we will use the De Giorgi "minimizing movement" approximation scheme to construct the gradient flow associated to Ent_{γ_d}.

Let $\mu^{(0)} = \mu_0 \in \mathbb{P}^*(\mathbb{R}^d)$ be given, and $\mu^{(1)} = \hat{\mu}$ obtained in Theorem 4.1. By induction, define step by step $\mu^{(n)}$ which realizes the minimum of

$$\mu \mapsto \frac{1}{2}W_2^2(\mu^{(n-1)},\mu) + \eta Ent_{\gamma_d}(\mu).$$

so we get a sequence of probability measures $\{\mu^{(n)}; n\geq 0\} \subset \mathbb{P}^*(\mathbb{R}^d)$.

Let $N = [\frac{1}{\eta}]$ be the integral part of $\frac{1}{\eta}$. Define

$$\nu_\eta(t,dx) = \sum_{k=1}^{N+1} \mu^{(k)}(dx)\mathbb{1}_{(t_{k-1},t_k]}(t), \quad \text{with } t_{N+1} = 1$$

Notice that $\nu_\eta(t,\cdot) \in Dom(\nabla Ent_{\gamma_d})$ for each $t > 0$.

Proposition 4.3 *The family $\{\nu_\eta(t,dx)dt; \eta > 0\}$ over $[0,1]\times\mathbb{R}^d$ is tight.*

Proof. By construction of $\{\mu^{(k)}; k\geq 1\}$, we have

$$\frac{1}{2}W_2^2(\mu^{(k-1)},\mu^{(k)}) + \eta Ent_{\gamma_d}(\mu^{(k)}) \leq \eta Ent_{\gamma_d}(\mu^{(k-1)}) \tag{4.54}$$

For any $1\leq q\leq N+1$, summing the above inequality from $k=1$ to q gives

$$\frac{1}{2}\sum_{k=1}^{q} W_2^2(\mu^{(k-1)},\mu^{(k)}) + \eta Ent_{\gamma_d}(\mu^{(q)}) \leq \eta Ent_{\gamma_d}(\mu^{(0)}).$$

For each $1 \leq q \leq N$,

$$W_2^2(\mu^{(0)}, \mu^{(q)}) \leq N\Sigma_{k=1}^N W_2^2(\mu^{(k-1)}, \mu^{(k)}) \leq 2N\eta Ent_{\gamma_d}(\mu^{(0)}) \leq 2Ent_{\gamma_d}(\mu^{(0)})$$

According to (4.54), we have

$$W_2^2(\mu^{(0)}, \mu^{(q)}) + Ent_{\gamma_d}(\mu^{(q)}) \leq 3Ent_{\gamma_d}(\mu^{(0)}) \tag{4.55}$$

Therefore the family $\{\mu^{(q)} : q \geq 0\}$ is tight: Let $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^d$ such that $\mu^{(q)}(K^c) < \varepsilon$, for $q \geq 0$. Now

$$\int_{[0,1]\times K^c} \nu_\eta(t, dx)dt = \sum_{k=1}^{N+1} \mu^{(k)}(K^c)(t_k - t_{k-1}) < \varepsilon$$

Therefore $\{\nu_\eta; \eta > 0\}$ is tight. □

Then there is a sequence $\eta \downarrow 0$ such that $\nu_\eta(t, dx)dt$ converges weakly to $\nu(dt, dx)$. Set $\mu^{(k)} = \rho^{(k)}\gamma_d$. Then

$$\nu_\eta(t, dx)dt = \left(\sum_{k=1}^{N+1} \rho^{(k)} 1_{(t_{k-1}, t_k]}(t)\right) d\gamma_d(x)dt = \rho_\eta(t, x)d\gamma_d(x)dt.$$

We have

$$\int_{[0,1]\times\mathbb{R}^d} \rho_\eta(t, x) \lg \rho_\eta(t, x)d\gamma_d(x)dt = \sum_{k=1}^{N+1} Ent_{\gamma_d}(\mu^{(k)})(t_k - t_{k-1}) \leq Ent_{\gamma_d}(\mu^{(0)}) < +\infty$$

Again using the lower semi-continuity of

$$\rho \mapsto Ent_{\gamma_d \otimes dt}(\rho),$$

we see that $Ent_{\gamma_d \otimes dt}(\nu) < +\infty$ and $\nu(dt, dx) = \rho(t, x)d\gamma_d(x)dt$, with

$$\int_{[0,1]\times\mathbb{R}^d} \rho(t, x) \lg \rho(t, x)d\gamma_d dt \leq Ent_{\gamma_d}(\mu^{(0)}).$$

It follows that for a.s. $t \in [0, 1]$, $Ent_{\gamma_d}(\rho(t, \cdot)) < +\infty$. Let

$$\nu_t(dx) = \rho(t, x)d\gamma_d(x).$$

By (4.55), $\sup_q m_2(\mu^{(q)}) < +\infty$. Then

$$\int_{[0,1]\times\mathbb{R}^d} |x|^2 \rho_\eta(t, x)d\gamma_d dt = \sum_{k=1}^{N+1} \left(\int_{\mathbb{R}^d} |x|^2 d\mu^{(k)}(x)\right)(t_k - t_{k-1}) \leq \sup_q m_2(\mu^{(q)}) < +\infty$$

Letting $\eta \downarrow 0$ in the above inequality, we get

$$\int_{[0,1]\times\mathbb{R}^d} |x|^2 \rho(t, x)d\gamma_d dt < +\infty$$

Therefore for a.s. $t \in [0, 1]$, $m_2(\nu_t) < +\infty$ and $\nu_t \in \mathbb{P}^*(\mathbb{R}^d)$.

Proposition 4.4 *The curve $\{\nu_t : t \in [0,1]\}$ solves the following Fokker-Planck equation*

$$-\int_{[0,1]\times\mathbb{R}^d} \alpha'(t)F\, d\nu_t dt + \int_{[0,1]\times\mathbb{R}^d} \alpha(t)LF\, d\nu_t dt = \alpha(0)\int_{\mathbb{R}^d} F d\mu_0 \tag{4.56}$$

for all $\alpha \in C_c^\infty([0,1)), F \in C_c^\infty(\mathbb{R}^d)$.

Proof. We have

$$\begin{aligned}\int_{[0,1]\times\mathbb{R}^d} \alpha'(t)F\, \nu_\eta(t,dx)dt &= \sum_{k=1}^{N+1} (\alpha(t_k) - \alpha(t_{k-1}))\int_{\mathbb{R}^d} F\rho^{(k)} d\gamma_d \\ &= \sum_{k=1}^{N} \alpha(t_k)\int_{\mathbb{R}}^d F\ (\rho^{(k)} - \rho^{(k-1)})d\gamma_d - \alpha(0)\int_{\mathbb{R}}^d F d\mu^{(1)}, \end{aligned}\tag{4.57}$$

since $\alpha(t_{N+1}) = \alpha(1) = 0$. On the other hand,

$$\begin{aligned}\int_{[0,1]\times\mathbb{R}^d} \alpha(t)LF\, \nu_\eta(t,dx)dt &= \sum_{k=1}^{N+1} \int_{t_{k-1}}^{t_k} \alpha(t)dt \int_{\mathbb{R}}^d LF\rho^{(k)} d\gamma_d \\ &= \sum_{k=0}^{N} \frac{1}{\eta}\int_{t_k}^{t_{k+1}} \alpha(t)dt\, \eta\int_{\mathbb{R}^d} LF\rho^{(k+1)} d\gamma_d. \end{aligned}\tag{4.58}$$

Let $\beta_k = \alpha(t_k) - \frac{1}{\eta}\int_{t_k}^{t_{k+1}} \alpha(t)dt$. Then combining (4.57) and (4.58),we have

$$\begin{aligned}&\int_{[0,1]\times\mathbb{R}^d} \alpha'(t)F\, \nu_\eta(t,dx)dt - \int_{[0,1]\times\mathbb{R}^d} \alpha(t)LF\, \nu_\eta(t,dx)dt \\ &= \sum_{k=1}^{N} \alpha(t_k)\Big[\int_{\mathbb{R}^d} F\ (\rho^{(k)} - \rho^{(k+1)})d\gamma_d - \eta\int_{\mathbb{R}^d} LF\rho^{(k+1)} d\gamma_d\Big] \\ &+ \sum_{k=1}^{N} \beta_k\eta\int_{\mathbb{R}^d} LF\rho^{(k+1)} d\gamma_d - \Big(\int_0^{t_1} \alpha(t)dt\Big)\int_{\mathbb{R}^d} LF\rho^{(1)} d\gamma_d \\ &-\alpha(0)\int_{\mathbb{R}^d} F\rho^{(1)} d\gamma_d. \end{aligned}\tag{4.59}$$

Note that $t_1 = \eta$ and $W_2^2(\mu_0, \mu^{(1)}) \leq \eta Ent_{\gamma_d}(\mu_0)$. Therefore, as $\eta \downarrow 0$, the sum of the last two terms tend to $-\alpha(0)\int_{\mathbb{R}^d} F d\mu_0$. By (4.53) in Theorem 4.1 and Proposition 4.1,

$$\eta\int_{\mathbb{R}^d} LF\rho^{(k+1)} d\gamma_d = \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle x - y, \nabla F(y)\rangle \pi^{(k)}(dx,dy),$$

where $\pi^{(k)} \in \mathscr{C}_0(\mu^{(k)}, \mu^{(k+1)})$ and $|\eta\int_{\mathbb{R}^d}\langle F, \rho^{(k+1)}\rangle d\gamma_d| \leq \|\nabla F\|_\infty W_2(\mu^{(k)}, \mu^{(k+1)})$. Note that $|\beta_k| \leq \|\alpha'\|_\infty \eta$ and

$$\begin{aligned}\sum_{k=1}^{N}\Big|\beta_k\eta\int_{\mathbb{R}^d} LF\rho^{(k+1)} d\gamma_d\Big| &\leq \|\alpha'\|_\infty\|\nabla F\|_\infty\eta\sum_{k=1}^{N} W_2(\mu^{(k)}, \mu^{(k+1)}) \\ &\leq \|\alpha'\|_\infty\|\nabla F\|_\infty\eta\sqrt{N}(\sum_{k=1}^{N} W_2^2(\mu^{(k)}, \mu^{(k+1)}))^{\frac{1}{2}} \\ &\leq \|\alpha'\|_\infty\|\nabla F\|_\infty\eta\sqrt{Ent_{\gamma_d}(\mu_0)} \to 0\ \ as\ \eta \downarrow 0. \end{aligned}$$

Set

$$I_k = \int_{\mathbb{R}^d} F\ (\rho^{(k)} - \rho^{(k+1)}) d\gamma_d - \eta \int_{\mathbb{R}^d} LF \rho^{(k+1)} d\gamma_d.$$

Using $\pi^{(k)} \in \mathscr{C}_0(\mu^{(k)}, \mu^{(k+1)})$, I_k can be expressed by

$$I_k = \int_{\mathbb{R}^d \times \mathbb{R}^d} (F(x) - F(y) - \langle x - y, \nabla F(y) \rangle) \pi^{(k)}(dx, dy).$$

Therefore

$$\begin{aligned}|I_k| &\leq \|\nabla^2 F\|_\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi^{(k)}(dx, dy) \\ &= \|\nabla^2 F\|_\infty\ W_2^2(\mu^{(k)}, \mu^{(k+1)}).\end{aligned}$$

So

$$\begin{aligned}\sum_{k=1}^{N} |\alpha(t_k) I_k| &\leq \|\alpha\|_\infty \|\nabla^2 F\|_\infty W_2^2(\mu^{(k)}, \mu^{(k+1)}) \\ &\leq \|\alpha\|_\infty \|\nabla^2 F\|_\infty \eta Ent(\mu_0) \to 0 \ \ as\ \eta \downarrow 0.\end{aligned}$$

Now letting $\eta \downarrow 0$ in (4.59), we get

$$\int_{[0,1] \times \mathbb{R}^d} \alpha'(t) F\, d\nu_t dt - \int_{[0,1] \times \mathbb{R}^d} \alpha(t) LF\, d\nu_t dt = -\alpha(0) \int_{\mathbb{R}^d} F d\mu_0.$$

□

In what follows, we will prove the existence of $\frac{d^o \nu_t}{dt}$ which satisfies that

$$\frac{d^o \nu_t}{dt} = -(\nabla Ent_{\gamma_d})(\nu_t).$$

Let $Z^{(k)} = (\nabla Ent_{\gamma_d})(\mu^{(k)})$ and define

$$Z_\eta(t, x) = \sum_{k=1}^{N+1} Z^{(k)}(x) \mathbb{1}_{(t_{k-1}, t_k]} \in \mathbb{R}^d.$$

Letting $T^{(k)} = I + \xi_k$, which pushes $\mu^{(k-1)}$ forward to $\mu^{(k)}$, we have

$$\begin{aligned}\int_{[0,1] \times \mathbb{R}^d} |Z_\eta(t, x)|^2 \nu_\eta(t, dx) dt &= \sum_{k=1}^{N+1} \int_{t_{k-1}}^{t_k} (\int_{\mathbb{R}^d} |Z^{(k)}|^2 d\mu^{(k)}) dt \\ &\leq \sum_{k=1}^{N+1} (t_k - t_{k-1}) \int_{\mathbb{R}^d} \frac{|\xi_k \circ (T^{(k)})^{-1}|}{\eta^2} d\mu^{(k)} \\ &\leq \frac{1}{\eta} \sum_{k=1}^{N+1} W_2^2(\mu^{(k-1)}, \mu^{(k)}) \leq 2 Ent_{\gamma_d}(\mu_0).\end{aligned}$$

Lemma 4.1 *There exists* $Z \in L^2(\mathbb{R}^d, \mathbb{R}^d; \mathbb{P}_\nu)$:

$$\int_{[0,1]\times\mathbb{R}^d} |Z_\eta(t,x)|^2 \, d\nu_t(dx)dt < +\infty$$

and a sequence $\eta \downarrow 0$ *such that*

$$\lim_{\eta\to 0} \int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), Z_\eta(t,x)\rangle \, \nu_\eta(t,dx)dt = \int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), Z(t,x)\rangle \, \nu_t(dx)dt$$

for all $\alpha \in C_c^\infty((0,1)), F \in C_c^\infty(\mathbb{R}^d)$.

Proof. Define a probability measure on $[0,1]\times\mathbb{R}^d\times\mathbb{R}^d$ by

$$\int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \psi(t,x,y)d\Gamma_\eta(t,x,y) = \int_{[0,1]\times\mathbb{R}^d} \psi(t,x,Z_\eta(t,x))\nu_\eta(t,dx)dt.$$

In another word,

$$\Gamma_\eta = (I\times Z_\eta)_*\mathbb{P}_{\nu_\eta},$$

where $I\times Z_\eta : (t,x) \mapsto (t,x,Z_\eta(t,x))$ and $\mathbb{P}_{\nu_\eta}(dt,dx) = \nu_\eta(t,dx)dt$. Then

$$(\pi_1,\pi_2)_*\Gamma_\eta = \mathbb{P}_{\nu_\eta}, (\pi_3)_*\Gamma_\eta = (Z_\eta)_*\mathbb{P}_{\nu_\eta}.$$

Note that $B_R = \{x||x| \le R\}$.

$$\begin{aligned}(\pi_3)_*\Gamma_\eta(B_R^c) &= \int_{[0,1]\times\mathbb{R}^d} \mathbb{1}_{B_R^c}(Z_\eta(t,x))\nu_\eta(t,dx)dt \\ &\le \frac{1}{R^2}\int_{[0,1]\times\mathbb{R}^d} |Z_\eta(t,x)|^2\nu_\eta(t,dx)dt \le \frac{2Ent_{\gamma_d}(\mu_0)}{R^2}.\end{aligned}$$

It follows that the family $\{(\pi_3)_*\Gamma_\eta : \eta > 0\}$ is tight; on the other hand, by Proposition 4.3, $\{\mathbb{P}_{\nu_\eta} : \eta > 0\}$ is tight. Therefore, the family $\{\Gamma_\eta : \eta > 0\}$ is tight. Up to a sequence, we get the weak convergence

$$(\pi_3)_*\Gamma_\eta \to w(dx) \quad \textit{and} \quad \Gamma_\eta \to \Gamma.$$

Then $(\pi_1,\pi_2)_*\Gamma = \rho(t,x)d\gamma_d dt, (\pi_3)_*\Gamma = w(dx)$ and

$$\int_{\mathbb{R}^d} |x|^2 w(dx) \le \liminf_{\eta\to 0}\int_{[0,1]\times\mathbb{R}^d} |Z_\eta(t,x)|^2 \, \nu_\eta(t,dx)dt \le 2Ent_{\gamma_d}(\mu_0);$$

hence $w \in \mathbb{P}_2(\mathbb{R}^d)$. Now by disintegration formula, there is a Borel family of probability $\Gamma_{t,x}(dy)$ in $\mathbb{R}^d : (t,x) \mapsto \int_{\mathbb{R}^d} f(y)\Gamma_{t,x}(dy)$ is Borel for $f \in \mathscr{B}(\mathbb{R}^d)$, such that

$$\int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \psi(t,x,y)d\Gamma(t,x,y) = \int_{[0,1]\times\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \psi(t,x,y)d\Gamma_{t,x}(y)\Big)d\nu_t(x)dt.$$

Define $Z(t,x) = \int_{\mathbb{R}^d} y d\Gamma_{t,x}(y)$. It is a Borel vector field. We have

$$\begin{aligned}\int_{[0,1]\times\mathbb{R}^d} |Z(t,x)|^2 \, d\nu_t(x)dt &\le \int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} |y|^2 d\Gamma(t,x,y) \\ &= \int_{\mathbb{R}^d} |y|^2 dw(y) \le 2Ent_{\gamma_d}(\mu_0) < +\infty.\end{aligned}$$

Now consider the function $(t,x,y) \mapsto \alpha(t)\langle \nabla F(x), y\rangle$, we have as $\eta \downarrow 0$

$$\int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), y\rangle d\Gamma_\eta(t,x,y) \to \int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), y\rangle d\Gamma(t,x,y); \qquad (4.60)$$

or

$$\int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), Z_\eta(t,x)\rangle \nu_\eta(t,dx)dt$$

tends to the right hand of (4.60). But

$$\begin{aligned}\int_{[0,1]\times\mathbb{R}^d\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), y\rangle d\Gamma(t,x,y) &= \int_{[0,1]\times\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \alpha(t)\langle \nabla F(x), y\rangle d\Gamma_{t,x}(y)\Big) d\nu_t(x)dt \\ &= \int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), Z(t,x)\rangle d\nu_t(x)dt.\end{aligned}$$

Note that the function $(t,x,y) \mapsto \alpha(t)\langle \nabla F(x), y\rangle$ is not bounded relative to y, however the passage to the limit in (4.60) can be verified by using the usual cut-off argument. □

Theorem 4.2 *The continuity equation*

$$\frac{d\nu_t}{dt} + \nabla\cdot(Z_t\nu_t) = 0 \qquad \textit{on } \ ]0,1[\times\mathbb{R}^d$$

holds.

Proof. The same computation as in the proof of Proposition 4.4 works. □

Theorem 4.3 *It holds that*

$$\frac{d^o\nu_t}{dt} = -(\nabla Ent_{\gamma_d})(\nu_t).$$

Proof. The continuity equation reads as

$$\int_{[0,1]\times\mathbb{R}^d} \alpha'(t)F(x)\, d\nu_t(x)dt + \int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F(x), Z_t(x)\rangle\, d\nu_t dt = 0.$$

For $\alpha \in C_c^\infty((0,1))$, the Fokker-Planck equation in Proposition 4.4 reads

$$-\int_{[0,1]\times\mathbb{R}^d} \alpha'(t)F\, d\nu_t dt + \int_{[0,1]\times\mathbb{R}^d} \alpha(t)LF\, d\nu_t dt = 0.$$

The two equations give

$$\int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F, Z_t\rangle\, d\nu_t dt = -\int_{[0,1]\times\mathbb{R}^d} \alpha(t)LF\, d\nu_t dt.$$

Let $\hat{Z} \in L^2(\mathbb{R}^d,\mathbb{R}^d,\mathbb{P}_\nu)$ be the orthogonal projection of Z on

$$\overline{\{\sum_i \beta_i\nabla\varphi_i : \beta_i \in C_c^\infty(]0,1[), \varphi_i \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mathbb{P}_\nu)}.$$

We have

$$\int_{[0,1]\times\mathbb{R}^d} \alpha(t)\langle \nabla F, \hat{Z}_t\rangle \, d\nu_t dt = -\int_{[0,1]\times\mathbb{R}^d} \alpha(t) LF \, d\nu_t dt.$$

Then there is a full measure subset $\Omega_F \subset [0,1]$ such that

$$\int_{\mathbb{R}^d} \langle \nabla F, \hat{Z}_t\rangle \, d\nu_t = -\int_{\mathbb{R}^d} LF \, d\nu_t, \qquad t \in \Omega_F.$$

Using the separability of $C_c^\infty(\mathbb{R}^d)$, there is a full measure subset $\Omega \subset [0,1]$ such that, for $t \in \Omega$,

$$\int_{\mathbb{R}^d} \langle \nabla F, \hat{Z}_t\rangle \, d\nu_t = -\int_{\mathbb{R}^d} LF \, d\nu_t, \qquad \forall F \in C_c^\infty(\mathbb{R}^d).$$

But by Proposition 4.2, $\int_{\mathbb{R}^d} LF \, d\nu_t = (\partial_{\nabla F} Ent_{\gamma_d})(\nu_t)$. It follows that $(\nabla Ent_{\gamma_d})(\nu_t)$ exists and $(\nabla Ent_{\gamma_d})(\nu_t) = -\hat{Z}_t = \frac{d^o \nu_t}{dt}$. □

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