Quadratic Covariation and Itô's Formula for a Bi-fractional Brownian Motion

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The 6th Workshop on Markov Processes and Related Topics

Motivation

$B^{H,K} = \{B_t^{H,K} : t \geq 0\}$: a bi-fractional Brownian motion with indices H, K such that $2HK = 1$ $(0 < H < 1, 0 < K \leq 1)$.

- THEN the usual quadratic variation $\left[B^{H,K}, B^{H,K}\right]_t$ equals to $2^{1-K}t$, that is

$$\left[B^{H,K}, B^{H,K}\right]_t = P - \lim_{n\to\infty} \sum_{j=1}^{n} \left(B_{t_j}^{H,K} - B_{t_{j-1}}^{H,K}\right)^2 = 2^{1-K}t,$$

where the limit is uniform in t and $t_j = \frac{jt}{n}$.

Motivation

- Quadratic covariation $[f(B^{H,K}), B^{H,K}]$ of $f(B^{H,K})$ and $B^{H,K}$:

$$[f(B^{H,K}), B^{H,K}] = ?$$

and

$$E|\,[f(B^{H,K}), B^{H,K}]\,| \le ?$$

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$$E|[f(B^{H,K}), B^{H,K}]| \leq ?$$

- This motivates the subject matter of the study!

Bifractional Brownian motion?

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- However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes.

Bifractional Brownian motion?

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- Therefore, it seems interesting to study the quadratic covariation and extension of Itô's formula of bifractional Brownian motion—a rather special class of self-similar Gaussian processes.

Bifractional Brownian motion?

- Consider the stochastic partial differential equations of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 W}{\partial t \partial x},$$

with initial condition $u(0, x) = 0$, where $W = \{W(t, x), t \geq 0, x \in \mathbb{R}\}$ is a two-parameter Wiener process.

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with initial condition $u(0, x) = 0$, where $W = \{W(t, x), t \geq 0, x \in \mathbb{R}\}$ is a two-parameter Wiener process.
  - Then the solution u equals to a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$, multiplied by the constant $(2\pi)^{\frac{1}{4}} 2^{-\frac{1}{8}}$.

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  - ⋆ an analogue of Föllmer-Protter-Shiryayev's formula.

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  - ⋆ $E\left[B_s^{H,K} B_t^{H,K}\right] = \frac{1}{2^K}\left[(t^{2H} + s^{2H})^K - |t-s|^{2HK}\right]$, $\forall s,t \geq 0$;

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- ⋆ Indices : $0 < H < 1$, $0 < K \leq 1$.

Notation: Bi-fractional Brownian motion (Bi-fBm)

⋆ THIS PROCESS was first introduced by Houdré and Villa (2002):

[1] C. Houdré and J. Villa, An example of infinite dimensional quasi-helix. *Stochastic models (Mexico City, 2002)*, 195-201, Contemp. Math., **336** (2003).

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- ⋆ THE PROCESS $B^{H,K}$ is strongly locally nondeterministic.
- ⋆ THE PROCESS $B^{H,K}$ has Hölder continuous paths of order $\alpha < HK$ and its paths are not differentiable.

Notation: Bi-fractional Brownian motion (Bi-fBm)

⋆ QUADRATIC VARIATION $[B^{H,K}, B^{H,K}]_t$ satisfies

$$[B^{H,K}, B^{H,K}]_t = \begin{cases} 0, & \text{if } \frac{1}{2} < HK < 1 \\ 2^{1-K}t, & \text{if } HK = \frac{1}{2} \\ +\infty, & \text{if } 0 < HK < \frac{1}{2} \end{cases}$$

for all $t > 0$.

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- ⋆ IF $HK = \frac{1}{2}$ AND $K \neq 1$ the process $B^{H,K}$ is a short-memory process.

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- ⋆ IF $HK > \frac{1}{2}$ the process $B^{H,K}$ has long memory, and for $HK < \frac{1}{2}$ it has short memory.
- ⋆ IF $HK = \frac{1}{2}$ AND $K \neq 1$ the process $B^{H,K}$ is a short-memory process.
- ⋆ FOR EVERY $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ AND $K \in (0, 1)$, the process $B^{H,K}$ is not a semimartingale .

Notation: Bi-fractional Brownian motion (Bi-fBm)

⋆ THE PROCESS $B^{H,K}$ satisfies the following estimates (see Houdré-Villa [1]) :

$$2^{-K}|t-s| \leq E\left[\left(B_t^{H,K} - B_s^{H,K}\right)^2\right] \leq 2^{1-K}|t-s|.$$

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⋆ THE LEFT ESTIMATE can improved as

$$|t-s|^{2HK} \leq E\left[\left(B_t^{H,K} - B_s^{H,K}\right)^2\right],$$

Notation: Bi-fractional Brownian motion (Bi-fBm)

⋆ by applying the inequality

$$(1+x)^\alpha \leq 1 + (2^\alpha - 1)x^\alpha, \quad 0 \leq x \leq 1$$

with $0 \leq \alpha \leq 1$.

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with $0 \le \alpha \le 1$.

- Remark:

$$(1+x)^{\alpha} \le 1+\alpha x^{\alpha} \le 1+x^{\alpha}.$$

Notation: Bi-fractional Brownian motion (Bi-fBm)

In the following discussion we assume that $2HK = 1$.

- ⋆ Denote $\mu = E(B_s^{H,K} B_r^{H,K})$ and $\rho^2 = sr - \mu^2$.

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In the following discussion we assume that $2HK = 1$.

- ⋆ Denote $\mu = E(B_s^{H,K} B_r^{H,K})$ and $\rho^2 = sr - \mu^2$.
- ⋆ THEN we have

$$0 \le r - \mu \le 2^{(1-K)/2}\sqrt{r(s-r)},$$

$$0 \le s - \mu \le 2^{(1-K)/2}\sqrt{s(s-r)},$$

for $s \ge r \ge 0$

Notation: Bi-fractional Brownian motion (Bi-fBm)

⋆ FOR $s \geq r \geq 0$ we have

$$r(s-r) \leq \rho^2 \leq 4^{1-K} s(s-r);$$

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⋆ For $s \geq r \geq 0$ we have

$$r(s-r) \leq \rho^2 \leq 4^{1-K} s(s-r);$$

⋆ For $T \geq t > s > t' > s' > 0$ we have

$$0 \leq E(B_t^{H,K} - B_s^{H,K})(B_{t'}^{H,K} - B_{s'}^{H,K}) \leq c \frac{(t-s)(t'-s')}{\sqrt{t'(t-t')}}.$$

Bi-fractional Brownian motion: References

★ [2] I. Kruk, F. Russo and C. A. Tudor, Wiener integrals, Malliavin calculus and covariance measure structure, *J. Funct. Anal.* **249** (2007), 92-142.

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Stochastic integral

♠ the stochastic integral $(2HK \geq 1)$

$$\int_0^t u_s dB_s^{H,K}$$

is of Skorohod type (see Es-sebaiy and Tudor [5]).

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* The Malliavin derivative $D^{H,K}$;
* The variance of integral ($2HK \geq 1$)

$$E\left|\int_0^T u_s dB_s^{H,K}\right|^2;$$

* Estimate the express

$$\frac{\partial^2}{\partial r \partial l} R(r,l).$$

where $R(s,r) = \frac{1}{2^K}\left[\left(s^{2H} + r^{2H}\right)^K - |s-r|^{2HK}\right]$.

Stochastic integral

- By applying the decomposition

$$R(r,l) = \frac{1}{2^K}\left[(s^{2H}+l^{2H})^K - (s^{2HK}+l^{2HK})\right] + \frac{1}{2^K}\left[-|s-l|^{2HK} + (s^{2HK}+l^{2HK})\right],$$

we can estimate the express

$$\left|\frac{\partial^2}{\partial r \partial l} R(r,l)\right|$$

Stochastic integral

- For $2HK = 1$ we have

$$\left|\frac{\partial^2}{\partial r \partial l} R(r,l)\right| = (2H-1)2^{-K}\left(r^{2H}+l^{2H}\right)^{K-2} r^{2H-1} l^{2H-1}$$
$$\leq (2H-1)2^{-K} r^{2H\beta(K-2)+2H-1} l^{2H\alpha(K-2)+2H-1}$$

by Young's inequality with $0 < \alpha < \frac{1}{2-K}, 1 > \beta > \frac{1-K}{2-K}$, $\alpha + \beta = 1$.

Results

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is of Skorohod type (see Es-sebaiy and Tudor [5]).

Result 1: Integration wrt local time

- $\mathscr{L}^{H,K}$: the local time of bi-fBm defined by Tanaka's formula

$$|B_t^{H,K}-x| = |B_0^{H,K}-x| + \int_0^t \mathrm{sign}(B_s^{H,K}-x)dB_s^{H,K} + \mathscr{L}^{H,K}(t,x);$$

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⋆ Occupation formula

$$\int_0^t f(B_s^{H,K},s)ds = \int_{\mathbb{R}} dx \int_0^t f(x,s)\mathscr{L}^{H,K}(x,ds);$$

Result 1: Integration wrt local time

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$$\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t).$$

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* We find a Banach Space $\mathscr{H}$ of measurable functions such that the above integral is well-defined for $f \in \mathscr{H}$, and

$$E\left|\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t)\right| \leq c\|f\|_{\mathscr{H}}.$$

Result 1: Integration wrt local time

- Similarly, we can define the integral of two parameters

$$\int_{\mathbb{R}} \int_0^t f(x,s) \mathscr{L}^{H,K}(dx,ds).$$

Result 2: Quadratic covariation

♠ Result II : We give the existence of quadratic covariation $[f(B^{H,K}), B^{H,K}]$ of $f(B^{H,K})$ and $B^{H,K}$:

$$[f(B^{H,K}),B^{H,K}]_t$$

$$\equiv P - \lim_{n\to\infty} \sum_{k=1}^{n} \{f(B^H_{t_k}) - f(B^H_{t_{k-1}})\}(B^H_{t_k} - B^H_{t_{k-1}})$$

with $t_k = kt/n$.

Result 2: Quadratic covariation

- Quadratic covariation $[f(B^{H,K},\cdot), B^{H,K}]$ of $f(B^{H,K},\cdot)$ and $B^{H,K}$:

$$[f(B^{H,K},\cdot), B^{H,K}]_t$$

$$\equiv P - \lim_{n\to\infty} \sum_{k=1}^{n} \{f(B^H_{t_k}, t_k) - f(B^H_{t_{k-1}}, t_{k-1})\}(B^H_{t_k} - B^H_{t_{k-1}})$$

with $t_k = kt/n$.

Result 2: Quadratic covariation

- The quadratic covariation can also be defined by the following limit in probability (See Russo *et al* (2000)):

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ f(B^{H,K}_{s+\varepsilon}) - f(B^{H,K}_s) \right\} (B^{H,K}_{s+\varepsilon} - B^{H,K}_s) ds.$$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \left\{ f(B^{H,K}_{s+\varepsilon}, s+\varepsilon) - f(B^{H,K}_s, s) \right\} (B^{H,K}_{s+\varepsilon} - B^{H,K}_s) ds.$$

Result 3: Two Itô formulas

♠ Result III. The following Bouleau-Yor's formula holds

$$F(B_t^{H,K}) = F(0) + \int_0^t f(B_s^{H,K}) dB_s^{H,K} - \frac{1}{2}\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t),$$

where $F' = f \in \mathscr{H}$.

Result 3: Two Itô formulas

♠ Result III. The following (Föllmer-Protter-Shiryayev's) formula holds

$$F(B_t^{H,K}) = F(0) + \int_0^t f(B_s^{H,K}) dB_s^{H,K} + 2^{K-2}[f(B^{H,K}), \quad B^{H,K}]_t,$$

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Result 3: Two Itô formulas

$$F(B_t^{H,K},t) = F(0,0) + \int_0^t f(B_s^{H,K},s)dB_s^{H,K} - \frac{1}{2}\int_{\mathbb{R}}\int_0^t f(x,s)\mathscr{L}^{H,K}(dx,ds)$$

$$F(B_t^{H,K},t) = F(0,0) + \int_0^t f(B_s^{H,K},s)dB_s^{H,K} + 2^{K-2}[f(B^{H,K},\cdot),B^{H,K}]_t,$$

Result 4: The local time on curve

♠ Result IV. Let $t \mapsto a(t)$ be a continuous function on $[0,1]$. Then the local time $\ell^{H,K}(a,t)$ of Bi-fBm $B^{H,K}$ on curve a exists for all $t \in [0,1]$, and

$$\ell^{H,K}(a,t) = -2^{1-K} \int_{\mathbb{R}} \int_0^t 1_{[a(s),+\infty)}(x) \mathscr{L}^{H,K}(dx, ds)$$

where $f_a(x,s) = 1_{[a(s),\infty)}(x)$.

A Banach Space of Measurable Functions

Consider the set $\mathscr{H}$ of measurable functions f on $\mathbb{R}$ such that $\|f\| < +\infty$, where

$$\|f\| = \sqrt{\int_0^1 \frac{ds}{\sqrt{2\pi s}} \int_{\mathbb{R}} f^2(x) e^{-\frac{x^2}{2s}} dx} + \int_0^1 \frac{ds}{s\sqrt{2\pi s}} \int_{\mathbb{R}} |f(x)x| e^{-\frac{x^2}{2s}} dx.$$

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$\Downarrow\Downarrow\Downarrow$

* $\mathscr{H}$ is a Banach space;
* the set $\mathscr{E}$ of elementary functions is dense in $\mathscr{H}$.

Integration wrt local time

Lemma (1)

For any $f_\triangle = \sum_j x_j 1_{(a_{j-1},a_j]} \in \mathscr{E}$, *the integral*

$$\int_{\mathbb{R}} f_\triangle(x)\mathscr{L}^{H,K}(dx,t) := \sum_j x_j \left[\mathscr{L}^{H,K}(a_j,t) - \mathscr{L}^{H,K}(a_{j-1},t)\right]$$

is well-defined, and

$$E\left|\int_{\mathbb{R}} f_\triangle(x)\mathscr{L}^{H,K}(dx,t)\right| \le c\|f_\triangle\|$$

for all $0 \le t \le T$.

Integration wrt local time

Now, for $f \in \mathscr{H}$ we can define

$$\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t) := \lim_{n\to\infty} \int_{\mathbb{R}} f_{\triangle,n}(x)\mathscr{L}^{H,K}(dx,t), \qquad \text{in } L^1,$$

if $f_{\triangle,n} \to f$ in $\mathscr{H}$, where $\{f_{\triangle,n}\} \subset \mathscr{E}$. Clearly, the definition is well-defined, and

$$E\left|\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t)\right| \leq c\|f\|.$$

Integration wrt local time

* For all $f \in C^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t) = -\int_{\mathbb{R}} f'(x)\mathscr{L}^{H,K}(x,t)dx, \qquad t \geq 0;$$

Integration wrt local time

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$$\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t) = -\int_{\mathbb{R}} f'(x)\mathscr{L}^{H,K}(x,t)dx, \qquad t \geq 0;$$

* Let $f, f_1, f_2, \ldots \in \mathscr{H}$ and let $f_n \to f$ in $\mathscr{H}$. We then have

$$\int_{\mathbb{R}} f_n(x)\mathscr{L}^{H,K}(dx,t) \longrightarrow \int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t), \qquad \text{in } L^1$$

for all $0 \leq t \leq T$, as $n \to \infty$.

Theorem (1)

Let the measurable function $f \in \mathscr{H}$ and let $F' = f \in \mathscr{H}$. Then the following Itô type formula holds:

$$F(B_t^{H,K}) = F(0) + \int_0^t f(B_s^{H,K})dB_s^{H,K} - \frac{1}{2}\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t).$$

p-variation of local time

Lemma (2)

For $t \geq 0, x \in \mathbb{R}$ *set*

$$\widehat{B}_t^{H,K}(x) := \int_0^t 1_{(B_s^{H,K} > x)} dB_s^{H,K}.$$

Then the estimate

$$E\left[\left(\widehat{B}_t^{H,K}(b) - \widehat{B}_t^{H,K}(a)\right)^2\right] \leqslant C_{H,K,t}(b-a)^{2-K} \tag{2.1}$$

holds for all $2HK = 1$ *and* $a, b \in \mathbb{R}, a < b$*, where* $C_{H,K,t} > 0$ *is a constant depending only on* H, K, t*.*

p-variation of local time

Theorem (2)

Let $2KH = 1$. Then the limit in probability

$$\lim_{|\Delta_n|\to 0} \sum_{a=a_0<a_1<\ldots<a_n=b} \left|\mathscr{L}^{H,K}(a_{i+1},t) - \mathscr{L}^{H,K}(a_i,t)\right|^{\frac{2}{2-K}}$$

exists, where $|\Delta_n| = \max_j\{|a_{j+1} - a_j|\}$.

p-variation of local time

Theorem (3)

Let $2HK = 1$. Then the local time $\mathscr{L}^{H,K}(x,t)$ is of bounded p-variation in x for any $0 \le t \le T$, for all $p > \frac{2}{2-K}$, almost surely.

p-variation of local time

Theorem (4)

For $2HK = 1$, if $x \mapsto f(x)$ is of bounded p-variation with $1 \leqslant p < \frac{2}{K}$, then the (Young) integral

$$\int_a^b f(x)\mathscr{L}^{H,K}(dx,t)$$

$$:= \lim_{|\Delta_n|\to 0} \sum_{a=a_0<a_1<\ldots<a_n=b} f(a_j)\left[\mathscr{L}^{H,K}(a_{j+1},t) - \mathscr{L}^{H,K}(a_j,t)\right]$$

is well defined.

Quadratic covariation

♠ Consider a partition $t_j = \frac{jt}{n}, j = 0, 1, 2, \ldots, n$ of $[0, t]$.

♠ Denote $\triangle_j B^{H,K} = B^{H,K}_{t_j} - B^{H,K}_{t_{j-1}}$, for $1 \le j \le n$. Then the quadratic covariation $\left[f(B^{H,K}), B^{H,K}\right]$ is defined by

$$\left[f(B^{H,K}), B^{H,K}\right]_t = \lim_{n\to\infty} \sum_{j=1}^{n} \{f(B^{H,K}_{t_j}) - f(B^{H,K}_{t_{j-1}})\}\triangle_j B^{H,K}$$

as a limit in probability.

Quadratic covariation

♠ A result introduced by Russo-Vallois yields

$$\left[f(B^{H,K}), B^{H,K}\right]_t = 2^{1-K}\int_0^t f'(B^{H,K}_s)ds$$

for all $f \in C^1(\mathbb{R})$.

$\Downarrow\Downarrow$

Quadratic covariation

♠ A result introduced by Russo-Vallois yields

$$[f(B^{H,K}), B^{H,K}]_t = 2^{1-K} \int_0^t f'(B_s^{H,K}) ds$$

for all $f \in C^1(\mathbb{R})$.

$\Downarrow\Downarrow$

* For $f \in C^1(\mathbb{R})$ we have

$$[f(B^{H,K}), B^{H,K}]_t = -2^{1-K} \int_{\mathbb{R}} f(x) \mathscr{L}^{H,K}(dx, t);$$

Quadratic covariation

♠ A result introduced by Russo-Vallois yields

$$[f(B^{H,K}), B^{H,K}]_t = 2^{1-K} \int_0^t f'(B_s^{H,K}) ds$$

for all $f \in C^1(\mathbb{R})$.

⇓⇓

∗ For $f \in C^1(\mathbb{R})$ we have

$$[f(B^{H,K}), B^{H,K}]_t = -2^{1-K} \int_{\mathbb{R}} f(x) \mathscr{L}^{H,K}(dx, t);$$

∗ For $f \in C(\mathbb{R})$ we have

$$\sum_{j=0}^{n-1} f(B_{t_j}^{H,K}) \left(B_{t_{j+1}}^{H,K} - B_{t_j}^{H,K} \right)^2 \xrightarrow{P} 2^{1-K} \int_0^t f(B_s^{H,K}) ds.$$

A related result

♠ If $p \geq 2$ is even and $f \in C(\mathbb{R})$, then

$$n^{\frac{p}{2}-1} \sum_{j=1}^{n} f(B_{t_j}^{H,K}) \left(\triangle_j B^{H,K}\right)^p \xrightarrow{P} 2^{1-K} c_p \int_0^t f(B_s^{H,K}) ds,$$

where c_p denotes the p-moment of a random variable $\xi \sim N(0,1)$.

An identity

Theorem (5)

Let $f \in \mathscr{H}$. *Then the quadratic covariation* $\left[f(B^{H,K}), B^{H,K}\right]_t$ *exists in* L^1, *and*

$$E\left|\left[f(B^{H,K}), B^{H,K}\right]_t\right| \leq c\|f\|$$

and

$$\left[f(B^{H,K}), B^{H,K}\right]_t = -2^{1-K}\int_{\mathbb{R}} f(x)\mathscr{L}^{H,K}(dx,t) \tag{3.1}$$

for all $t \in [0,T]$.

Föllmer-Protter-Shiryayev's formula

♠ Let the measurable function $f \in \mathscr{H}$ and let $F' = f \in \mathscr{H}$. Then the following Itô type formula holds:

$$F(B_t^{H,K}) = F(0) + \int_0^t f(B_s^{H,K}) dB_s^{H,K} + 2^{K-2}\left[f(B^{H,K}), B^{H,K}\right]_t .$$

♠ Let $(x,s) \mapsto f(x,s)$ a measurable function on $\mathbb{R} \times [0,T]$.

♠ In this section, we define the integral for two parameters

$$\int_{\mathbb{R}} \int_0^t f(x,s) \mathscr{L}^{H,K}(dx,ds), \quad t \geq 0, \tag{4.1}$$

and study existence of the quadratic covariation $\left[f(B^{H,K},\cdot), B^{H,K}\right]$.

Weighted quadratic covariation

♠ $\left[f(B^{H,K}), B^{H,K}\right]_t = 0$ if $2HK > 1$.

Weighted quadratic covariation

♠ $\left[f(B^{H,K}), B^{H,K}\right]_t = 0$ if $2HK > 1$.

♠ Weighted quadratic covariation $\left[f(B^{H,K}), B^{H,K}\right]^{(W)}$:

$$\left[f(B^{H,K}), B^{H,K}\right]_t^{(W)}$$

$$\equiv P - \lim_{n\to\infty} \sum_{k=1}^{n} (k-1)^{2HK-1} \{f(B_{t_k}^{H,K}) - f(B_{t_{k-1}}^{H,K})\} \triangle_k B^{H,K}$$

with $t_k = kt/n$.

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THANKS !