The 6^{th} Workshop on Markov Processes
and Related Topics

Asymptotic optimal investment for minimizing ruin probability

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1 Motivation

- The estimation of ruin probabilities is a central topic in risk theory. It is known that if the claim sizes have exponential moments (i.e. so-called small claim case), the ruin probability decrease exponentially with the initial surplus, see, for instance, Asmussen[1]. But when there is a stochastic return on investment, the situation may be different.
- Kalashnikov and Norberg[12] have investigated the problem under the additional assumption that all the surplus is invested in the risky market, likewise did by Paulsen and Gjessing [13], Frovola, Kabanov and Pergamenshchikov[14]. In all of these cases it was shown that, even if the claim sizes are small claim size, the ruin probability decreases only with some negative power of the initial surplus. Thus, for large capital, investing more than the surplus into the risky market can not be optimal.

Thus, one problem is : if an insurer has the opportunity to invest in a risky market, what is the minimal ruin probability she can obtain? In particularly, can she do better than keeping the funds in the bond? And if yes, how much can she do better?

- In classical risk model or perturbed classical risk model, the ruin probability (denoted by $\tilde{\Psi}(x)$) has the following exponential upper bound
$$\tilde{\Psi}(x) \le \mathrm{e}^{-Rx}, \tag{1.1}$$
where x is the initial surplus and R is constant, namely, *adjustment coefficient*.
- Whether there are constant R such that the aforementioned upper bound holds for the minimal ruin probability (so-called value function, denoted by $\Psi(x)$) under the optimal investment strategy? Of course, there always is the possibility not to invest at all, resulting in an exponential bound for the ruin probability $\Psi(x)$.
- We want to find the optimal (i.e. the largest) coefficient R such that (1.1) holds true for $\Psi(x)$. That is to say we want to find the tightest upper bound for our value function.

It turn out that R is determined by a similar equation defining the Lundburg adjustment coefficient. The trading strategy corresponding to this optimal R is to invest constant amount of surplus in the risky market, independent of the current level of the surplus. It is shown that this constant strategy is asymptotically optimal, respectively asymptotically unique, in the sense that every “asymptotically different ” Markovian strategy yields an exponentially worse decay of the ruin probability.

2 Model and assumptions

- In this paper, the surplus process is given by

$$S_t = u + pt + \Pi_t - X_t + \sigma W_t. \tag{2.2}$$

- It is assumed that $\Pi_t = \sum_{i=1}^{N_S^1(t)} P_i, \quad X_t = \sum_{i=1}^{N_S^2(t)} C_i$, where $N_S^1(t),\ N_S^2(t)$ are two Poisson processes with parameters $\lambda_1,\ \lambda_2$. $\{P_i,\ i \geq 1\}$, $\{C_i,\ i \geq 1\}$ are two sequences of i.i.d positive random variables.

- Usually, it is assumed that $\{X_t,\ t \geq 0\}$ is the thinning process of $\{\Pi_t,\ t \geq 0\}$. For simplicity (without losing generality), $\{\Pi_t,\ t \geq 0\}$ and $\{X_t,\ t \geq 0\}$ are assumed to be independent in this paper. $\{W_t,\ t \geq 0\}$ is a standard Brownian Motion and is independent with $\{\Pi_t,\ t \geq 0\}$ and $\{X_t, t \geq 0\}$, σ is a positive constant.

Assumption on financial market

- We assume that the standard assumptions of continuous-time financial models hold, that is
  1. continuous trading;
  2. no transaction cost or tax is involved in trading;
  3. all assets are infinitely divisible.
- The price of the risky asset is assumed to follow the stochastic differential equation

$$\frac{\mathrm{d}P(t)}{P(t)} = \mathrm{d}Z_t = \alpha \mathrm{d}t + \beta \mathrm{d}B_t,\ \alpha > 0,\ \beta > 0,\ t \geq 0. \qquad (2.3)$$

  Here $\{B_t,\ t \geq 0\}$ is a standard Brownian Motion, and Process $S = \{S_t,\ t \geq 0\}$ and Process $Z = \{Z_t,\ t \geq 0\}$ are assumed to be independent.
- Denote by $\boldsymbol{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the smallest filtration satisfying the usual condition such that the process $S = \{S_t, t \geq 0\}$ and $Z = \{Z_t, t \geq 0\}$ are measurable.

Assumptions on investment strategies

- Let $\{A_t\}$ denote the amount invested into risky market at time t, we allow that the company invests more than its current surplus into risk market.
- The strategies $\{A_t,\ t \geq 0\}$ have to be predictable w.r.t. $\mathcal{F}_t$ and the admissible set is

$$\mathcal{A}=\Big\{ A = (A_t)_{t\geq 0}\text{: } A \text{ is predictable and } \mathsf{P}\left[\int_0^t A^2(s)\mathrm{d}s < \infty\right] \text{ for all } t \in [0,\infty)\Big\}.$$

- In this paper, we focus on the Markov control, i.e.

$$A_t = A(Y_{t-}^{A,b}), \tag{2.4}$$

where $A(\cdot)$ is called the *defining function* of the Markov strategy A_t.

3 Problem and main results

- The dynamic of the risk process of the insurer with such investment strategy is given by

$$\begin{aligned} \mathrm{d}Y_t^{x,A} &= [\alpha A_t + p]\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}\Pi_t + A_t\beta \mathrm{d}B_t - \mathrm{d}X_t, \\ Y_0^{x,A} &= x. \end{aligned} \tag{3.5}$$

The time of ruin with initial surplus x and strategy A is

$$\tau(x, A) = \inf\{t \geq 0 : \ Y_t^{x,A} < 0\} \tag{3.6}$$

and corresponding ruin probability is

$$\Psi(x, A) = \mathsf{P}(\tau(x, A) < \infty). \tag{3.7}$$

The value function is

$$\Psi(x) = \inf_{A \in \mathcal{A}} \Psi(x, A). \tag{3.8}$$

- Let $R(A)$ be the solution to equation

$$\frac{1}{2}(\sigma^2+A^2\beta^2)r^2+\lambda_1[M_P(-r)-1]+\lambda_2[M_C(r)-1]-(p+\alpha A)r=0. \tag{3.9}$$

This is the Lundburg exponent in the case of a constant strategy A. The Lundburg exponent for our problem is $R=\sup R(A)$.

- One can finally find that R is the solution to

$$\frac{1}{2}\sigma^2 r^2+\lambda_1 M_P(-r)+\lambda_2 M_C(r)=\lambda_1+\lambda_2+cr+\frac{1}{2}\frac{\alpha^2}{\beta^2} \tag{3.10}$$

and the optimal constant strategy is $A^*=\dfrac{\alpha}{\beta^2 R}$.

- **Lemma 1.** *Condition on the aforementioned R and A^*, $M_t := \exp\{-RY_t^{x,A^*}\}$ is a martingale and*

$$\Psi(x) = \inf_{A \in \mathcal{A}} \Psi(x, A) \leq \Psi(x, A^*) \leq \mathrm{e}^{-Rx}. \tag{3.11}$$

- **Definition** Let $0 < r < r_\infty$ be given. We say that C has a *uniform exponential moment in the tail distribution for* r, if the following condition holds:

$$\sup_{y \geq 0} \mathsf{E}\left[\mathrm{e}^{-r(y-C)} \middle| C > y\right] < \infty. \tag{3.12}$$

Lemma 2. Assume that C has a uniform exponential moment in the tail distribution for R. Then for each $A \in \mathcal{A}$, the process $(\exp\{-RY_t^{x,A}\})_{t\geq 0}$ is a uniformly integrable submartingale.

Lemma 3. For arbitrary $A \in \mathcal{A}$ and $x \in (0, +\infty)$, the surplus process $Y_t^{x,A}$ converges almost surely on $\{\tau(x, A) = \infty\}$ to ∞ for $t \to \infty$.

Theorem 4. Assume that C has a uniform exponential moment in the tail distribution for R. Then the ruin probability satisfies , for every admissible control process $A(t) \in \mathcal{A}$,

$$\Psi(x, A) \geq L\mathrm{e}^{-Rx}, \tag{3.13}$$

where

$$L = \inf_{y \geq 0} \frac{\int_y^\infty \mathrm{d}G_C(u)}{\int_y^\infty \mathrm{e}^{-R(y-u)}\mathrm{d}G_C(u)} = \frac{1}{\sup\limits_{y\geq 0} \mathsf{E}[\mathrm{e}^{-r(y-C)}|C > y]} > 0. \tag{3.14}$$

Lemma 5. Under the conditions of Theorem 4, if there exists $\epsilon > 0$ and $x_\epsilon \geq 0$ such that

$$|A(x) - A^*| \geq \epsilon \text{ for all } x \geq x_\epsilon, \tag{3.15}$$

then there exist r_ϵ and A_ϵ such that

$$\Psi(x, A) \geq A_\epsilon \mathrm{e}^{-r_\epsilon x}. \tag{3.16}$$

Theorem 6. Under the conditions of Theorem 4, Let $\tilde{A}(\cdot)$ be the defining function of the optimal investment strategy $\tilde{A}$. If this function has a limit for $x \to \infty$, then this limit is given by

$$\lim_{x\to\infty} \tilde{A}(x) = A^*. \tag{3.17}$$

Remark 1. What can we say from Theorem 6? One can find that when the surplus x tends to ∞, the optimal strategy tends to choose the constant strategy A^*. Obviously, when the surplus is very large, such strategy is a very conservative investment strategy. So from Theorem 6 we know that minimizing the ruin probability is an extremely conservative approach for the insurers.

References

[1] Asmussen, S. Ruin Probabilities. World Scientific, Singapore, 2000.

[2] Browne, S. Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of Ruin, *Mathmatics of Operation Research*, **20**(1995), 937-958.

[3] Dufersne, F. and Gerber, H.U. Risk theory for the compound Poisson risk process that is perturbed by diffusion, *Insurance: Mathematics and Economics*, **10**(1991), 51-59.

[4] Flemming, W.H. and Soner, H.M. Controlled Markov processes and viscosity solutions, springer-verlag, NewYork, 1993.

[5] J. Gaier, P. Grandits and Schachermayer, W. Asymptotic ruin probability and optimal investment, *Annals of Applied Probability* , **13(3)**, (2003), 1054-1076.

[6] Gerber,H. An Introduction to Mathematical Risk Theory, Richard D, Irwin, 1979.

[7] Gerber, H. and Shiu, E.S.W. Optimal dividends: analysis with Brownian motion *North American Actuarial Journal*, **8(1)**(2004), 1-20.

[8] Hipp, C. and Schmidli, H. Asymptotics of ruin probabilities for controlled risk processes in the small claims case, *Scandinavian Actuarial Journal*, **5**(2004), 321-335.

[9] Hipp, C. and Plum, M. Optimal investment for insurers, *Insurance: Mathematics and Economics*, **27** (2000), 215-228.

[10] Hipp, C. and Plum, M. Optimal investment for investors with state dependent income, and for insurers, *Finance and Stochastics*, **7**(2003), 299-321.

[11] Hipp, C. and Taksar, M. Stochastic control for optimal new business, *Insurance: mathematics and economics*, **26**(2000), 185-192.

[12] Kalashnikov, V. and Norberg, R. Power tailed ruin Probabilities in the presence of risky investments, *Stochastic Processes and Their Applications*, **98**(2002), 211-228.

[13] Paulsen, J. and Gjessing, H. K. Ruin theory with stochastic return on investment, *Adv. in Appl. Probab.*, **29**(1997), 965-985.

[14] Frovola, A. G., Kabanov, Y. M. and Pergamenshchikov, S. M. In the insurance business risky investment are dangerous, *Finance and Stochastics,* **6**(2002), 227-235.

[15] Melnikov, L. Risk analysis in Fianace and Insurance. CRC Press: New York, 2004.

[16] Protter, P. E. Stochastic Integration and Differential Equations, second edn, Springe-Verlag, Berlin, 2004.

[17] Schmidli, H. On minimizing the ruin probability by investment and reinsurance, *Annals of Applied Probability*, **12**(2002), 890-907.

[18] Yang, H. L. and Zhang, L. H. Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics*, **37**(2005), 615-634.

Thanks!