Time change and Feller measures

Jiangang Ying

July 23, 2008

The talk is based on a joint paper with Ping He:

Revuz measure under time change, Sci China A, vol 51, no 3(2008), 321-328

Background

- Douglas integral:

D: unite disk on $\mathbf{R}^2$, f: C^1-function on ∂D

$$\frac{1}{2}\int_D |\nabla Hf(x)|^2 \, dx = \frac{1}{2}\int_{\partial D\times\partial D\setminus d} \frac{(f(\xi)-f(\eta))^2}{4\pi(1-\cos(\xi-\eta))} d\xi d\eta,$$

where Hf denotes the harmonic function on the planar unit disk D with boundary value f, or H is Poisson kernel.

Douglas, J.(1931). *Solution of the problem of Plateau,* Trans. Amer. Math. Soc., **33**, 263-321

• left= the Dirichlet integral of Hf, which is the trace of Dirichlet form of BM $B = (B_t)$ on ∂D.

Trace? Roughly speaking, the Dirichlet form of time change.

ϕ: a local time of BM on ∂D (ϕ increases as long as BM hits ∂D)

$\tau = (\tau_t)$: the right continuous inverse of ϕ.

$X_t = B(\tau_t)$, time change, which is a symmetric Markov process on ∂D. Its Dirichlet form is the trace.

• right= an energy integral of f on ∂D of pure jump with jump kernel

$$\frac{1}{4\pi(1-\cos(\xi-\eta))}.$$

To re-state the Douglas integral formula,

(1) trace of $\mathbf{D}$ on ∂D is a pure jump form with the kernel as above.

(2) the jump kernel of time change of BM is the kernel on the right.

• Task in this talk.

To find the jump measure of time change of a Markov process in a more general framework.

• Approach: Probabilistic potential theory related to Markov processes.

semigroup, resolvent, excessive measure, energy functional, Revuz measure, Lévy system, exit system, ...

Historical Remark

1930, J. Douglas, **TAMS**: Brownnian motion on disk

1962, J.L. Doob, **Ann Inst Fourier**: Brownian motion on a domain

1964, M. Fukushima, **Nagoya J Math**: Brownian motion, Naim kernel=Feller kernel

1975, Y. LeJan: Dirichlet forms, purely analytic

2004, Fukushima, He, Ying, **Annals of Probability**: For symmetric diffusions, time changed jump measure = Feller measure,

2006, Chen, Fukushima, Ying, **Annals of Probability**: For symmetric Markov processes, time changed jump measure= Feller measure + original jump measure

Additive functionals

Additive functionals are defined through shift operator θ_t.

$A = (A_t : t \geq 0)$: an adapted right continuous real process

(1) positive: $A_t \geq 0$,

(2) $A_{t+s} = A_s + A_t \circ \theta_s$ a.s. for any $t, s \geq 0$.

PCAF: a continuous additive functional.

Example: $A_t = t$, occupation time, local time, ... are all PCAF's.

Time change

$$X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \mathsf{P}^x)$$

a right Markov process on the state space $(E, \mathscr{E})$ with transition semigroup (P_t), cemetery point ∂ and life time ζ.

$A = (A_t)$: positive continuous additive functional of X;

$\tau_t := \inf\{s > 0 : A_s > t\}$, right continuous inverse,

$R_A := \inf\{t > 0 : A_t > 0\}$, first increasing time,

$F := \{x : \mathsf{P}^x(R_A = 0) = 1\}$, fine support of A, denoted by $\mathrm{supp}(A)$

$T := \inf\{t > 0 : X_t \in F\} = R_A$,

$\tilde{X}_t := X(\tau_t)$: a Markov process on F, time change of X induced by A.

- Potential operator of X: the mean time of X staying in a set

$$Uf(x) := \mathsf{E}^x \int_0^\infty f(X_t)\mathrm{d}t.$$

potential operator along A,

$$U_A f(x) := \mathsf{E}^x \int_0^\infty f(X_t) dA_t = \int_T^\infty f(X_t) dA_t.$$

- potential operator of $\tilde{X}$

$$\tilde{U} f(x) := \mathsf{E}^x \int_0^\infty f(\tilde{X}_t) dt.$$

Lemma $U_A = P_F \tilde{U}$, *where* P_F *is the balayage kernel on* F,

$$P_F(x, A) := \mathsf{P}^x(X_T \in A, T < \infty).$$

- h is excessive for $X \Rightarrow h|_F$ is excessive for $\tilde{X}$.

- h is excessive for $\tilde{X} \Rightarrow P_F h$ is excessive for X.

Revuz measure

- Given $m \in \text{Exc}(X)$. For any H: PCAF

$$\xi_H^m(f) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathsf{E}^m \int_0^t f(X_t) dH_t,$$

Revuz measure of H relative to m.

Revuz correspondence: $H \mapsto \xi_H^m$.

(Getoor: If a measure ξ does not charge semi-polar sets, then there exists a unique PCAF H such that $\xi_H^m = \xi$.)

For one-domensional BM, only empty sets are semipolar. Hence there exists a PCAF $L^x = (L_t^x)$ such that $\xi_L^m = \delta_x$ for any real x. L^x is called the local time at x.

- $\tilde{m} := \xi_A^m$. Then $\tilde{m} \in \text{Exc}(\tilde{X})$.

Energy functionals

The energy functional of X, defined on $\mathbf{S}(X) \times \mathrm{Exc}(X)$,

$$L(m,h) := \lim_{t \downarrow 0} \frac{1}{t} \langle m, h - P_t h \rangle,$$

due to P.A. Meyer.

Roughly speaking

$$L(m,h) = \int -G(h)\, dm,$$

where G is the infinitesimal generator of X in some sense. This leads to the energy integral in classical situation.

Theorem (Meyer) $\xi_H^m(f) = L(m, U_H f)$. *(when m is dissipative)*

Theorem $L(m, P_F h) = \tilde{L}(\tilde{m}, h)$ *for* $h \in \mathbf{S}(\tilde{X})$.

• Invariance of Revuz measure

A PCAF H is carried by F if supp$(H) \subset F$.

If H is carried by F, then $t \mapsto H(\tau_t)$ is continuous and hence $H_\tau = (H(\tau_t))$ is a PCAF of $\tilde{X}$, denoted simply by $\tilde{H}$. Naturally $\xi_H^m = \tilde{\xi}_{\tilde{H}}^{\tilde{m}}$ denotes the Revuz of $\tilde{H}$ relative to $\tilde{m}$ computed in $\tilde{X}$.

Combining two theorems above we would have Invariance of Revuz measure

Theorem *If H is carried by F, then*

$$\xi_H^m = \tilde{\xi}_{\tilde{H}}^{\tilde{m}}.$$

Note that the left side is independent of A.

Lévy system(Ikeda, Watanabe)

(N,H): To characterize the jumps of X, there exist a kernel N and a PCAF H, such that for any $f \geq 0$ on $E \times E$ vanishing on diagonal, the dual predictable projection of the increasing process $t \mapsto \sum_{s \leq t} f(X_{s-}, X_s)$ is

$$\int_0^t Nf(X_s)dH_s.$$

where $Nf := \int_E N(\cdot, dy)f(\cdot, y)dy$.

In the case of Lévy processes, let $H_t = t$ and $N(x, dy) = J(dy - x)$, where J is the Lévy measure. Then (N,H) is the Lévy system.

Lévy system is not unique. However the jump measure defined below, which combines N and H together, is unique as m is given.

Jump measure

For $m \in \mathsf{Exc}(X)$, define the jump measure of X relative to m by

$$J_X^m(f) = \lim_{t \to 0} \frac{1}{t} \mathsf{E}^m \sum_{s \le t} f(X_{s-}, X_s),$$

which is a σ-finite measure on $E \times E \backslash d$.

Using Lévy system, we have

$$J_X^m(dx, dy) = \xi_H^m \cdot N(x, dy).$$

Exit system

A beautiful but much more complicated notion, originated from K. Itô's excursion theory, then contributed by Getoor, Sharpe, Motoo, Maisonneuve......

Lévy system may be viewed as a special case of Exit system.

Exit system describes excursions away from a homogeneous set.

M: the closure of $\{t \geq 0 : X_t \in F\}$ (or a homogeneous set);

$T_t := T \circ \theta_t + t$;

$G := \{t > 0 : T_t > T_{t-} = t\}$, left end-points of excursion intervals;

Recall Itô's excursion theory, the excursion process of 1-dim BM away from zero is a Poisson point process whose intensity measure is a σ-finite measure on Ω.

An exit system is a pair $(\widehat{\mathsf{P}}, L)$ of a kernel $\widehat{\mathsf{P}}$ from $(\Omega, \mathscr{F}^0)$ to $(E, \mathscr{E})$ and a positive (not necessarily continuous) additive functional L of X such that

$$\mathsf{E}^x \sum_{s \in G} Z_s \cdot f \circ \theta_s = \mathsf{E}^x \int_0^\infty Z_s \widehat{\mathsf{P}}^{X_s}(f) dL_s, \qquad (1)$$

for $x \in E$, any positive random variable f on Ω, and any positive optional process $Z = (Z_s)$.

The existence is due to Maisonneuve.

- Assumption. L is continuous and carried by F.

- when does it hold?

(1) F is closed and finely perfect;

(2) X is in weak duality and Hunt hypothesis H: semipolar is m-polar.

How to find a Lévy system for time change process $\tilde{X}$?

Set $\tilde{N}(x, \mathrm{d}y) := 1_F(x)N(x, \mathrm{d}y)1_F(y)$, $K_t := 1_F(X_t)\mathrm{d}H_t$ and $U(x, dy) := \hat{\mathsf{P}}^x(X_T \in \mathrm{d}y)$ for $x \in F, y \in F_\partial$. Since both K and L are CAF's carried by F, they are τ-continuous. Hence $\tilde{K} := K_\tau$ and $\tilde{L} := L_\tau$ are PCAF's of $\tilde{X}$.

Theorem *Lévy system of $\tilde{X}$,*

$$\mathsf{E}^x \sum_{s \le t} f(\tilde{X}_{s-}, \tilde{X}_s) = \mathsf{E}^x \int_0^t \tilde{N} f(\tilde{X}_s) \mathrm{d}\tilde{K}_s + \mathsf{E}^x \int_0^t U f(\tilde{X}_t) \mathrm{d}\tilde{L}_s.$$

This means the Lévy system of $\tilde{X}$ consist of two parts, one from the original Lévy system and the other from exit system of F.

• It is now easy to compute $J_{\tilde{X}}^{\tilde{m}}$, the jump measure of $\tilde{X}$ relative to $\tilde{m}$.

Theorem $J_{\tilde{X}}^{\tilde{m}} = 1_{F\times F} \cdot J_X^m + U$, *where* U *is the Feller measure of* X *on* F*, defined by*

$$U(dx, dy) = \xi_L^m(\mathrm{d}x) \cdot U(x, \mathrm{d}y) 1_F(y).$$

The first part comes from the jump of X and the second part from excursions of X away from F.

Thank You