Decompositions and structural analysis of stationary infinitely divisible processes

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Outline

1. General ID processes
2. Classes of ID processes
3. Stationary selfdecomposable processes

By a natural analogy to Gaussian processes we consider

Definition

Let T be an arbitrary nonempty set. A process $\mathbf{X} = \{X_t\}_{t \in T}$ is said to be an infinitely divisible (ID) stochastic process if for any $t_1, \dots, t_n \in T$ the random vector

$$(X_{t_1}, \dots, X_{t_n})$$

has an ID distribution.

Examples:

(i) Lévy processes. (Additive processes.)

(ii) Linearly additive random fields.
Recall, a random field $\{X_t\}_{t \in \mathbb{R}^d}$ is called linearly additive if the stochastic process ξ defined by $\xi_s = X_{a+sb}$, $s \in \mathbb{R}$ has independent increments for every pair (a, b), $a, b \in \mathbb{R}^d$. Mori (1992) characterized all ID, stochastically continuous, linearly additive random fields. (Chentsov type representations.)

(iii) Multiparameter Lévy processes.

(iv) ID random measures. E.g., cluster Poisson processes, Cox processes with infinitely divisible directing measures, etc.

(v) Gaussian and stable processes.

(vi) Stationary ID processes. E.g., moving averages driven by Lévy processes (Ornstein-Uhlenbeck processes), harmonizable processes, etc.

(vii) Stationary increment ID processes. E.g., fractional processes.

(viii) Linear ID random fields. $\{X_t\}_{t \in H}$, H is a linear space (Hilbert space, $\mathcal{M}(\mathbb{R}^d)$, etc.),

$$X_{ax+by} = aX_x + bX_y \quad a.s.$$

for every $x, y \in H$, $a, b \in \mathbb{R}$. (Cylindrical processes.)

In his fundamental work Maruyama (1970) defined a Lévy measure of an ID process on a special σ-ring of subsets of $\mathbb{R}^T$. Such σ-ring has a complicated structure when the index set T is uncountable.

We simplify this approach defining a (path) Lévy measure on the cylindrical σ-field

$$(\mathcal{B}(\mathbb{R}))^T = \prod_{t \in T} \mathcal{B}(R_t) \qquad (R_t = \mathbb{R})$$

of $\mathbb{R}^T$ for an arbitrary index set T.

Definition

A measure ν on the cylindrical σ-field $(\mathcal{B}(\mathbb{R}))^T$ is said to be a Lévy measure on $\mathbb{R}^T$ if

(i) for every $t \in T$

$$\int_{\mathbb{R}^T} \left(|x_t|^2 \wedge 1\right) \nu(dx) < \infty$$

and

(ii) for every countable set $T_1 \subset T$ such that $\nu\left\{x : x_{|T_1} \equiv 0\right\} > 0$ there exists $t \notin T_1$ such that

$$\nu\left\{x : x_{|T_1} \equiv 0,\ x_t \neq 0\right\} > 0.$$

Remark

(a) *If T is a countable index set, then condition (ii) in the definition of a Lévy measure is equivalent to*

$$\nu\{\mathbf{0}\} = 0, \tag{1}$$

where $\{\mathbf{0}\}$ denotes the origin of $\mathbb{R}^T$.

(b) *When T is uncountable, (1) does not make sense. We introduce (ii) as a proper generalization of (1) to arbitrary index sets. It gives the* <u>*uniqueness*</u> *of a Lévy measure corresponding to an infinitely divisible process.*

(c) *ν is a σ-finite Lévy measure if and only if for some countable set $T_0 \subset T$*

$$\nu\left\{x : x_{\big| T_0} \equiv 0\right\} = 0. \tag{2}$$

Lévy-Khintchine representation

Notation: Define the truncation function

$$[\![u]\!] := \frac{u}{|u| \vee 1} = \begin{cases} u & |u| \leq 1, \\ 1 & u > 1, \\ -1 & u < -1, \end{cases}$$

and truncation of $x \in \mathbb{R}^T$ by applying the truncation function to each component x_t

$$[\![x]\!]_t := [\![x_t]\!], \quad t \in T.$$

$$\mathbb{R}^{(T)} = \{x \in \mathbb{R}^T : x_t = 0 \text{ for all but finitely many } t\}.$$

$$\langle y, x \rangle = \sum_{t \in T} y_t x_t, \quad y \in \mathbb{R}^{(T)},\ x \in \mathbb{R}^T.$$

Theorem (Lévy-Khintchine representation)

Let $\mathbf{X} = \{X_t\}_{t\in T}$ *be an infinitely divisible stochastic process. Then there exist a unique (generating) triplet* (Σ, ν, b) *consisting of*

(i) *a nonnegative symmetric operator* $\Sigma : \mathbb{R}^{(T)} \mapsto \mathbb{R}^T$,

(ii) *a Lévy measure* ν *on* $\mathbb{R}^T$,

(iii) *a function* $b \in \mathbb{R}^T$,

such that for any $y \in \mathbb{R}^{(T)}$

$$\mathbb{E}e^{i\sum_{t\in T} y_t X_t} =$$

$$\exp\left\{-\frac{1}{2}\langle y, \Sigma y\rangle + \int_{\mathbb{R}^T} \left(e^{i\langle y,x\rangle} - 1 - i\langle y, [\![x]\!]\rangle\right)\nu(dx) + i\langle y, b\rangle\right\}.$$

Lévy-Itô representation

Any infinitely divisible process $\mathbf{X} = \{X_t\}_{t \in T}$ has a version $\mathbf{X}' = \{X'_t\}_{t \in T}$ of the form

$$X'_t = G_t + \int_{\mathbb{R}^T} x_t \left(N(dx) - \frac{\nu(dx)}{|x_t| \vee 1} \right) + b_t,$$

where $\mathbf{G} = \{G_t\}_{t \in T}$ is a mean-zero Gaussian process, N is a Poisson random measure on $\mathbb{R}^T$ with intensity ν, and $\mathbf{G}$ and N are independent.

The independence of Gaussian and Poissonian parts allows to study these parts separately. We will concentrate on ID processes without Gaussian part.

EXAMPLES OF PATH LÉVY MEASURES:

1. Lévy processes.

$\mathbf{X} = \{X_t\}_{t \geq 0}$ be a Lévy process with

$$\mathbb{E} e^{iuX_t} = e^{t\psi(u)},$$

$$\psi(u) = \int_{-\infty}^{\infty} (e^{iuv} - 1 - iu[\![v]\!])\, \eta(dv).$$

Here $T = \mathbb{R}_+$. What is the path Lévy measure ν of $\mathbf{X}$?

ANSWER: Path Lévy measure ν of a Lévy process $\mathbf{X}$ is the image measure of $\eta \times \mathrm{Leb}$ by

$$\mathbb{R} \times \mathbb{R}_+ \ni (v, s) \mapsto v\mathbf{1}_{[s,\infty)} \in \mathbb{R}^{\mathbb{R}_+}.$$

In particular, every such ν is concentrated on the set of one-step functions

$$S = \text{supp}\ \nu = \{v\mathbf{1}_{[s,\infty)} : v \in \mathbb{R},\ s \geq 0\}.$$

(Precisely, $\nu_*(\mathbb{R}^{\mathbb{R}_+} \setminus S) = 0$.) *Properties such as discontinuities of sample path of Lévy processes are inherited from the support of the path Lévy measure.*

For a Poisson process with parameter λ,

$$\text{supp}\ \nu = \{\mathbf{1}_{[s,\infty)} : s \geq 0\}$$

and ν is the image measure of $\eta \times \text{Leb}$ by the map $s \mapsto \mathbf{1}_{[s,\infty)}$.

2. ID point processes.

Let N be an ID point process on $\mathbb{R}^d$. Thus,

$$\{N(A) : A \in \mathcal{B}_0(\mathbb{R}^d)\}$$

is an ID process indexed by bounded Borel subsets of R^d. Its Lévy measure ν is obtained on the cylindrical σ-field of $R^{\mathcal{B}_0(\mathbb{R}^d)}$.

It can be shown that ν is concentrated on $\mathbf{N}_{\mathbb{R}^d}$, the space of nonnegative integer-valued measures, finite on bounded Borel sets.

The restriction of ν to $\mathbf{N}_{\mathbb{R}^d}$ is known as KLM measure of N.

We begin with the case $T = \{1, \dots, n\}$.

Definition (Class $ID_\rho(\mathbb{R}^n)$)

Let ρ be a given Lévy measure on $(0, \infty)$. An ID distribution μ on $\mathbb{R}^n$ belongs to $ID_\rho(\mathbb{R}^n)$ if its Lévy measure ν is of the form

$$\nu(A) = \eta_\rho(A) := \int_0^\infty \eta(As^{-1})\,\rho(ds), \quad A \in \mathcal{B}(\mathbb{R}^n),$$

for some measure η on $\mathbb{R}^n$ with $\eta\{0\} = 0$.

EXAMPLES OF $ID_\rho(\mathbb{R}^n)$ classes:

1. α-stable distributions: $\rho(ds) = s^{-\alpha-1}\,ds$, $\alpha \in (0,2)$;
2. selfdecomposable distributions: $\rho(ds) = s^{-1}\mathbf{1}_{(0,1]}(s)\,ds$;
3. tempered α-stable distributions: $\rho(ds) = s^{-\alpha-1}e^{-s}\,ds$, $\alpha \in (0,2)$;
4. Goldie–Steutel–Bondesson class: $\rho(ds) = e^{-s}\,ds$;
5. Thorin class (generalized gamma convolutions): $\rho(ds) = s^{-1}e^{-s}\,ds$;
6. type G (conditionally Gaussian) distributions: $\rho(ds) = e^{-s^2/2}\,ds$;
7. Maejima class: $\rho(ds) = s^{-1}e^{-s^2/2}\,ds$;
8. Jurek class: $\rho(ds) = \mathbf{1}_{(0,1]}(s)\,ds$.
9. general ID: $\rho(ds) = \delta_1(ds)$.

Define

$$\Upsilon_\rho(\eta) = \eta_\rho = \int_0^\infty \eta(\cdot\, s^{-1})\, \rho(ds)$$

on the domain

$\mathrm{dom}\Upsilon_\rho^{(d)} = \{\eta : \ \eta \text{ measure on } \mathbb{R}^d,\ \eta\{0\} = 0, \text{ and } \eta_\rho \text{ Lévy measure}\}.$

$\mathrm{dom}\Upsilon_\rho^{(d)}$ is a dense subset of the set of Lévy measures. It coincides with the set of all Lévy measures if and only if

$$\int_0^\infty s^2 \vee 1\, \rho(ds) < \infty.$$

$$ID_{\rho_2}(\mathbb{R}^n) \subset ID_{\rho_1}(\mathbb{R}^n) \iff \exists\, \rho : \ \rho_2 = \rho_1 \circledast \rho.$$

Ref.: Barndorff-Nielsen, R., Thorbjörnsen.
General Υ-transformations, ALEA 2008.

Definition

A stochastic process $\mathbf{X} = \{X_t\}_{t \in T}$ is an ID_ρ-process if for every $n \geq 1,\ t_1, \ldots, t_n \in T$,

$$\mathcal{L}\{(X_{t_1}, \ldots, X_{t_n})\}$$

is a probability measure in the class $ID_\rho(\mathbb{R}^n)$.

A stable process is an ID_ρ-process with $\rho(ds) = s^{-\alpha-1}\, ds$. Stationary stable processes and random fields have been studied extensively by many authors, including Hardin, Samorodnitsky, Pipiras – Taqqu, and R.

We will concentrate on selfdecomposable (SD) processes. That is, ID_ρ-processes for $\rho(ds) = s^{-1}\mathbf{1}_{(0,1]}(s)\, ds$.

3. Stationary selfdecomposable processes

Let $T = \mathbb{R}^d$ or $\mathbb{Z}^d$. Recall that a process $\mathbf{X} = \{X_t\}_{t \in T}$ is stationary if $\forall\, t_1, \ldots, t_n, t \in T$

$$(X_{t_1+t}, \ldots, X_{t_n+t}) \stackrel{d}{=} (X_{t_1}, \ldots, X_{t_n}).$$

Theorem (Stationary SD processes)

Let $\{X_t\}_{t \in T}$ be a stationary measurable mean-zero selfdecomposable process. Then there exists a Borel space $(S, \mathcal{B}(S))$ equipped with a σ-finite measure m and a measurable measure m-preserving flow

$$\phi_t : S \mapsto S \qquad t \in T$$

such that $\forall\ t \in T$

$$X_t = \int_S f(\phi_t(s))\, M(ds) \qquad a.s. \tag{3}$$

Theorem (Stationary SD processes (cont.))

The stochastic integral is with respect to an independently scattered random measure M on $(S, \mathcal{B}(S))$ satisfying

$$\mathbb{E}e^{iuM(A)} = \exp\{m(A)\psi(u)\}, \quad A \in \mathcal{B}(S), \tag{4}$$

with

$$\begin{aligned}\psi(u) &= \int_0^1 (e^{ius} - 1 - ius)s^{-1}\, ds \\ &= -\gamma + \mathrm{Ci}(u) - \ln(u) + i(\mathrm{Si}(u) - u).\end{aligned}$$

Here Ci, Si *are the cosine and sine integral functions, respectively, and* γ *is the Euler constant.*

Stationary stable processes: R. (AoP 1995, 2001).

Decomposition of stationary SD processes:

$$X_t = \int_S f(\phi_t(s))\, M(ds).$$

Suppose $S = A \cup B$, where A, B disjoint shift invariant Borel sets in S. Define

$$X_t^A = \int_A f \circ \phi_t \, dM, \qquad X_t^B = \int_B f \circ \phi_t \, dM.$$

Then $\{X_t^A\}_{t \in T}$ and $\{X_t^B\}_{t \in T}$ are independent mean zero stationary SD processes.

Flow ϕ	Stationary SD process $\mathbf{X}$
decomposition into invariant parts	sum of indep. stat. SD components

Decomposition of $S = \mathbb{R}^{\mathbb{Z}}$ with respect to the shift θ:

- $C_0 = \{s \in S : \theta s = s\}$;
- C_1 is the largest modulo m shift-invariant set disjoint with C_0 such that $m_{|C_1}$ is equivalent to a probability measure;
- C_2 is the largest modulo m shift-invariant set disjoint with $C_0 \cup C_1$ such that $\forall A \subset C_2$, if $m(A) > 0$ then $\limsup_{n\to\infty} m(A \cap \theta^{-n}A) > 0$;
- C_3 is the largest modulo m shift-invariant set disjoint with $C_0 \cup C_1 \cup C_2$ such that C_3 does not contain a wandering set and $\forall A \subset C_3$, if $m(A) \in (0, \infty)$ then $\lim_{n\to\infty} m(A \cap \theta^{-n}A) = 0$;
- C_4 the largest modulo m set in S such that $C_4 = \bigcup_{n\in\mathbb{Z}} \theta^{-n}W$ with $\theta^{-n}W$ disjoint. θ is dissipative on C_4.

The decomposition: $S = \bigcup_{i=0}^{4} C_i$ into disjoint shift invariant sets is obtained by combining the following decompositions in ergodic theory: Hopf decomposition, Krengel - Sucheston decomposition, and positive-null decomposition.

$$X_n = \sum_{i=0}^{4} X_n^{(i)}, \quad n \in \mathbb{Z},$$

where

$$X_n^{(i)} := \int_{C_i} f \circ \theta^n \, dM, \quad i = 0, \dots, 4.$$

Theorem

Every stationary zero mean selfdecomposable process $\{X_n\}_{n\in\mathbb{Z}}$ *can be written uniquely in distribution as the sum*

$$X_n = \sum_{i=0}^{4} X_n^{(i)}, \quad n \in \mathbb{Z},$$

where $\{X_n^{(i)}\}_{n\in\mathbb{Z}}$, $i = 0, \dots, 4$ *are independent stationary zero mean selfdecomposable process (some may be zero) such that*

(0) $\{X_n^{(0)}\}_{n\in\mathbb{Z}}$ *has constant paths;*

(1) $\{X_n^{(1)}\}_{n\in\mathbb{Z}}$ *is not ergodic;*

(2) $\{X_n^{(2)}\}_{n\in\mathbb{Z}}$ *is weakly mixing (and so ergodic) but not mixing;*

(3) $\{X_n^{(3)}\}_{n\in\mathbb{Z}}$ *is mixing and does not have mixed moving average component;*

(4) $\{X_n^{(4)}\}_{n\in\mathbb{Z}}$ *is a mixed moving average process.*

For an illustration, consider a bilateral integer-valued Markov chain $\{\xi_n\}_{n\in\mathbb{Z}}$ defined on the canonical coordinate space $S = \mathbb{Z}^{\mathbb{Z}}$. Suppose that $\{\xi_n\}_{n\in\mathbb{Z}}$ is irreducible and recurrent. Let Q^x be the distribution of $\{\xi_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}$ starting from $x \in \mathbb{Z}$ (i.e., $Q^x\{\xi_0 = x\} = 1$). Define a shift-invariant measure m on $\mathbb{Z}^{\mathbb{Z}}$ by

$$m(A) = \int_{\mathbb{Z}} Q^x(A)\, m_0(dx), \quad A \in \mathcal{B}(S),$$

where m_0 is a (possibly infinite) stationary distribution of the Markov chain.

Let M be an SD random measure with control measure m on $\mathbb{Z}^{\mathbb{Z}}$ given by (4) and let

$$X_n = \int_{\mathbb{Z}^{\mathbb{Z}}} f(\theta^n(s))\, M(ds)$$

be the corresponding stationary zero-mean SD process. Here $\int (f^2 \wedge |f|)\, dm < \infty$ and θ is the shift.
Then

the SD process $\{X_n\}_{n\in\mathbb{Z}}$ is mixing if and only if the Markov chain $\{\xi_n\}_{n\in\mathbb{Z}}$ is null-reccurent. In this case $\mathbb{Z}^{\mathbb{Z}} = C_3 \mod m$.

$\{X_n\}_{n\in\mathbb{Z}}$ is not ergodic when $\{\xi_n\}_{n\in\mathbb{Z}}$ is positive-recurrent. In this case $\mathbb{Z}^{\mathbb{Z}} = C_2 \mod m$.

Lévy processes representation. Suppose m as above is an infinite atomless measure on $\mathbb{R}^{\mathbb{Z}}$. Since measure spaces $(\mathbb{R}^{\mathbb{Z}}, m)$ and $(\mathbb{R}_+, Leb)$ are isomorphic, we can write

$$X_n = \int_0^\infty g(V^n(t))\, dZ_t, \quad n \in \mathbb{Z}$$

where $V : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a Lebesgue measure preserving transformation of $\mathbb{R}_+$ and Z_t is a Lévy process with

$$\mathbb{E}e^{iuX_t} = e^{t\psi(u)},$$

where $\psi(u) = -\gamma + \mathrm{Ci}(u) - \ln(u) + i(\mathrm{Si}(u) - u)$, as on page 20. Thus $\{X_n\}$ can be viewed as a process in the first order chaos of $Z_t,\ t \geq 0$.

D. Nualart and Schoutens (2000) gave a chaotic decomposition of $L^2(\Omega, \sigma(Z_t, t \geq 0), \mathbb{P})$ as

$$\bigoplus_{n=0}^{\infty} \bigoplus_{i_1,\ldots,i_n \in \mathbf{N}} \mathcal{H}^{(i_1,\ldots,i_n)},$$

where $\mathcal{H}^{(i_1,\ldots,i_n)}$ are spaces of multiple stochastic integrals with respect to strongly orthogonal Teugels martingales $Y_t^{(i)}$, $t \geq 0$. Such martingales are obtained by applying orthogonal polynomials to powers of jumps of Z_t, $t \geq 0$.

Orthogonal polynomials related to a selfdecomposable Lévy process Z_t, $t \geq 0$ can be given explicitely. These are orthogonal polynomials of $L^2([0,1], x\mathrm{d}x)$,

$$p_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k+1}{n} x^k .$$

$p_n(x) = P_n^{(0,1)}(2x-1) \leftarrow$ Jacobi polynomial.

$$p_0(x) = 1 \qquad p_1(x) = 3x - 2$$
$$p_2(x) = 10x^2 - 12x + 3 \qquad p_3(x) = 35x^3 - 60x^2 + 30x - 4$$
$$p_4(x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$$

$$\int_0^1 p_n(x)^2 \, x\mathrm{d}x = \frac{1}{2(n+1)} .$$

$\{\sqrt{2(n+1)}\, p_n : n \geq 0\}$ is a CONS for $L^2([0,1], x\mathrm{d}x)$.

Transformation V of $\mathbb{R}$, corresponding to the shift on $\mathbb{R}^{\mathbb{Z}}$, generates an isometry on each chaos space $\mathcal{H}^{(i_1,\ldots,i_n)}$. Ergodic decomposition of V induces related ergodic decompositions in the space of chaos of selfdecomposable processes.

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Thank you!