QSDs and Domain of Attraction Problem

From Markov Chains to Diffusions

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Suppose also that an asymptotic analysis of the process has revealed the certainty of eventual absorption at 0, but that explicit evaluation of the distribution of $X(t)$ for all $t \geq 0$ is unwieldy or even impossible.

Naturally, one then looks for characteristics of the process that, on the one hand, give more detailed information than the bare fact that eventual absorption is certain, and, on the other hand, are easier to obtain than the distribution of $X(t)$ for all $t \geq 0$.

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For in that case the process relaxes to the quasi-limiting regime after a relatively short time, and then, after a very much longer period, absorption will eventually occur.

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Interesting examples of this phenomenon can be found in the chemical literature, see Dambrine and Moreau (1981), Parsons and Pollett (1987) and the references mentioned there.

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Let $\mathbf{P_i}(\cdot) = \mathbf{P}(\cdot \mid \mathbf{X_0} = \mathbf{i})$ and If A is a finite measure on $\mathbb{N}$, let $\mathbf{P_A} = \sum \mathbf{a_i P_i}$. Here and below any unqualified sum is taken over $\mathbb{N}$.

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Finally, suppose that $\mathbb{N}$ is irreducible and that 0 is accessible from some (and hence from every) state in $\mathbb{N}$.

Introduction

We further define

$$T = \inf\{t \geq 0 : X(t) = 0\}$$

the absorption (hitting) time at 0. We shall only be interested in processes for which $\mathbf{E_i T} < \infty$ for all $i \geq 1$.

Review

A **Quasi-Stationary Distribution (QSD)** $M = (m_i)$ is a probability measure on $\{1, 2, \cdots\}$ with the property that, starting with $M = (m_i)$, the conditional distribution, given the event that at time t the process has not been absorbed, still $M = (m_i)$. That is,

$$\mathbf{P_M(X(t) = j|T > t) = \frac{\sum m_i P_i(X(t) = j)}{\sum m_i P_i(X(t) \neq 0)} = m_j.} \qquad (1)$$

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It is not hard to show if such M exist then

$$\mathbf{P_M(T > t) = e^{-\mu t}}$$

for some $\mu \in (0, \infty)$.

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Actually, there are a great deal of papers (nearly 400 papers, see P.K.Pollett "Quasi stationary distributions: a bibliography". available at http://www.maths.uq.edu.au/ pkp/papers/qsds/qsds.html, regularly undated) dealing with the **QSD**s

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Given $\mu \geq 0$ we call a measure M on $\mathbb{N}$ a **μ-invariant measure for Q** if for each $j \geq 1$,

$$\sum m_i q_{ij} = -\mu m_j, \tag{7}$$

and if for all $t > 0$,

$$\sum m_i p_{ij}(t) = e^{-\mu t} m_j, \tag{8}$$

it is called **μ-invariant on $\{1, 2, \cdots\}$ for P**.

Review

We call $M = (m_j)$ **A- the limit conditional distribution (A-LCD)** if A is a probability measure on $\{1, 2, \cdots\}$ and each $j \geq 1$

$$m_j = \lim_{t \to \infty} \mathbf{P_A}(\mathbf{X_t} = \mathbf{j} \mid \mathbf{T} > \mathbf{t}) \qquad (9)$$

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The A-LCD is a **QSD** (Vere-Jones(1996)).

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(ii) solve the domian of attraction problem, namely, characterize all probability measure A such that a given **QSD** M is a A-LCD.

Although (i) has been addressed for several cases, details about (ii) are known only for finite Markov processes, and for the subcritical Markov Branching Process(MBP).

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For the subcritical MBP, the decay parameter is $\lambda = (1-m)a$ where m is the per capita mean number of offspring. Also, N is λ-recurrent, and the λ-invariant measure (for Q and P) is finite which , after normalization, has the p.g.f.

$$Q(s) = 1 - exp\left(-\int_0^s \frac{1-m}{f(v)-v}dv\right).$$

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$$Q(s) = 1 - exp\left(-\int_0^s \frac{1-m}{f(v)-v} dv\right).$$

This gives the LCD even though N can be λ-null (iff $\Sigma p_j \log^+ j = \infty$).

Review

Furthermore, $Q_\delta(s) = 1 - (1 - Q(s))^\delta$ is a p.g.f. when $0 < \delta \leq 1$, and only then, and its weights comprise a μ-invariant measure for $\mu = \delta\lambda$. When $\delta < 1$ it is a A-LCD iff A has an upper tail which is regularly varying with index$-\delta$. When $\delta = 1$ the corresponding condition is regular variation with index -1 or that $\Sigma j a_j < \infty$. This describes all $A - LCD's$. See Asmussen and Hering (1983), p.122, for proofs.

Review

The **QSD** structure of birth and death processes has received much attention; See Van Doorn (1991) for references. In particular he shows that either the forward system has a unique solution and either no **QSD** exists ($\lambda = 0$) or there is a continuum of **QSD**'s indexed by μ in an interval $(0, \lambda]$; or the forward system is not uniquely solved and then $\lambda > 0$ and there is exactly one **QSD** . The λ-invariant measure is finite and, normalized, is the LCD. I'll back to these more details.

New results

We now discuss the **QSD** problem for a general Markov Chain.

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P.A. Ferrari, H. Kesten, S. Martinez and P. Picco (1995) prove the following interesting result which makes no reference to this general theory. They make the following definition of *asymptotic remoteness* (AR) of the absorbing state: For each $t > 0$

$$\lim_{i \to \infty} \mathbf{P_i(T > t) = 1}. \tag{13}$$

New Results

Suppose that AR condition holds, Ferrari et al. prove that a **QSD** exists iff

$$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty \tag{14}$$

for some $\epsilon > 0$ and $i \in \mathbb{N}$.

New Results

Suppose that AR condition holds, Ferrari et al. prove that a **QSD** exists iff

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Indeed this condition is necessary with, or without, AR condition.

New Results

Suppose that AR condition holds, Ferrari et al. prove that a **QSD** exists iff

$$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty \tag{16}$$

for some $\epsilon > 0$ and $i \in \mathbb{N}$.

Indeed this condition is necessary with, or without, AR condition.

T. G. Pakes (1994) investigates what happens in a number of examples when AR condition fails. In fact, he examines quite closely two examples which violate AR condition but which nevertheless can have a **QSD** , showing AR condition is far from being a necessary condition, though it seems essential for the proofs of Ferrari et al.'s theorem.

New Results

Phil Pollett and Hanjun Zhang (2007) obtained that the following condition

$$\lim_{i \to \infty} \mathbf{E_i T} = \infty \tag{17}$$

can substitute for the AR condtion (that is, for each $t > 0$ $\lim_{i\to\infty} \mathbf{P_i}(\mathbf{T} > \mathbf{t}) = \mathbf{1}$.) which preserves the main result of Ferrari et al. (1995). We call condition (17) is AR* condition. That is, we have proved that if AR* holds, then a **QSD** exists iff

$$\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty$$

for some $\epsilon > 0$ and $i \in \mathbb{N}$.

New Results

Our main result is as follows

Theorem 1 Suppose that Q is stable, conservative and regular, and that Q restricted to $\{1, 2, \cdots\}$ is irreducible. Suppose further that $\mathbf{E_i}(\mathbf{e}^{\epsilon \mathbf{T}}) < \infty$. for some $\epsilon > 0$ and $i \in \mathbb{N}$, $\lim_{i \to \infty} \mathbf{E_i T} = \infty$, and that $\mathbf{P_i}(\mathbf{T} < \infty) = \mathbf{1}$ for some (and hence all) i.

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Then (i) $\lambda_C = \sup\{\lambda : E_i e^{\lambda T} < \infty\} > 0;$

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Then (i) $\lambda_C = \sup\{\lambda : E_i e^{\lambda T} < \infty\} > 0$;

(ii) for each $0 < \delta \leq \lambda_C$, there exists a **QSD** M such that

$$P_M(T > t) = e^{-\delta t}$$

That means that there is a continuum of **QSD**'s indexed by δ in an interval $(0, \lambda_C]$.

Sketch of the main proof

The first step: we prove that for any $0 < \delta \leq \lambda_C$, there exists a probability measure M such that
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$\delta = \sup\{\lambda : E_M e^{\lambda T} < \infty\}$.

In our proof, AR* holds, i.e. $\lim_{i \to \infty} \mathbf{E_i T} = \infty$ seems essential for the first step.

Sketch of the main proof

The second step: we prove that there exists a probability measure M_∞ such that

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It is analogous to Ferrari et al.(1995) for the second step. The method is based on the study of the renewal process with interarrival times distributed as the absorption time of the Markov Process with a given initial measure ν.

Sketch of the main proof

The third step: we prove that there exists a **QSD** M such that

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The third step: we prove that there exists a **QSD** M such that

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This step is the same as Proposition 4.1 in Ferrari et al. (1995) that is , the result follows from an application of fixed point theorem.

Birth and death process

We shall adopt the usual notation in prescribing birth rates $\lambda_i > 0 \ (i \geq 1)$, with $\lambda_0 = 0$, and death rates $\mu_i > 0 \ (i \geq 1)$. Now define by $\pi_1 = 1$ and

$$\pi_n = \prod_{k=2}^{n} \frac{\lambda_{k-1}}{\mu_k}, \qquad n \geq 2.$$

We will assume the process is absorbed with probability 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. \tag{18}$$

Birth and death process

In order to state our another result, we need the following notation:

$$S = \sum_{n=1}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{m=n+1}^{\infty} \pi_m.$$

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$$S = \sum_{n=1}^{\infty} (\lambda_n \pi_n)^{-1} \sum_{m=n+1}^{\infty} \pi_m.$$

As mentioned before, by Van Doorn (1991), if forward system is not uniquely solved, which is equivalent to $S < \infty$, then $\lambda_C > 0$ and there is exactly one **QSD**.

Birth and death process

Now our new result is
Theorem 2 For birth and death process, if both

$$\sum_{n=1}^{\infty} \frac{1}{\pi_n \lambda_n} = \infty. \tag{19}$$

and $S < \infty$ hold, then for any probability measure A , the unique **QSD** M is a $A - LCD$.That is, for any probability measure A

$$m_j = \lim_{t \to \infty} P_A(X_t = j \mid T > t)$$

exists and is a probability measure.

One-dimensional Diffusions

Consider the diffusion operator

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \quad on\ [0, \infty).$$

with $a \in C^2((0,\infty)), b \in C^1((0,\infty))$. Assume that $a(0) = 0, b(0) = 0$, which entails that 0 is an absorbing state; further assume $a(x) > 0$ for any $x > 0$, and denote $C(x) = \int_1^x \frac{b(t)}{a(t)} dt$.

One-dimensional Diffusions

Suppose that the diffusion is an absorption with probability 1, that is,

$$\int_1^{\infty} exp(-C(x))dx = \infty. \tag{20}$$

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Suppose that the diffusion is an absorption with probability 1, that is,

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We suppose that

$$\int_1^\infty exp(-C(x)) \int_x^\infty \frac{e^{C(x)}}{a(x)} dx < \infty \tag{23}$$

One-dimensional Diffusions

Now our result is

Theorem 3 For the diffusion process, suppose that $\int_1^\infty exp(-C(x))dx = \infty$hold, then (i) there is exactly one **QSD** iff $\int_1^\infty exp(-C(x)) \int_x^\infty \frac{e^{C(x)}}{a(x)}dx < \infty$ holds,
and (ii) when it holds, then for any probability measure on $(0,\infty)$ A , the unique **QSD** M is a $A-LCD$ in the sense that for any probability measure on $(0,\infty)$ A

$$M(B) = \lim_{t\to\infty} P_A(X_t \in B \mid T > t), \forall \text{ Borel set } B \subseteq (0,\infty)$$

exists and is a probability measure.

Further research

We wish to solve the domain of attraction problem for birth and death process. That is, we try to solve the case $S = \infty$. My guess result is the following If A a probability measure on N, and $x = sup\{\lambda : E_A e^{\lambda T} < \infty\}$ then

$$\lim_{t\to\infty} P_A(X_t = j \mid T > t) = q_j(x) \equiv \mu_1^{-1}\pi_j x Q_j(x), \quad j = 1, 2, \cdots ,$$

where $\{Q_n(x)\}$ is a system of polynomials recurrently defined by

$$\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x)Q_n(x) - \mu_n Q_{n-1}(x), n = 2, 3, 4, \cdots ,$$

$$\lambda_1 Q_2(x) = \lambda_1 + \mu_1 - x, \quad Q_1(x) = 1.$$

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