Log-Sobolev Inequalities on Metric Spaces

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1. Introduction:

- Some background on geometry of metric spaces:

M. Gromov ect.: metric spaces with negative sectional curvature

Bakry-Emery curvature condition: for diffusion processes

G. Perelmen's papers on Poincaré Conjecture: Using some results of geometry on metric space with a measure.

Since then, geometry on metric measure spaces was studied by many authors, for example:

J. Lott, K.T. Sturm, S.I. Ohta, F. Otto, C. Villani, Allice Chang.

The key points are to extend all important results on Riemannian manifolds to metric spaces.

- The new definition of Ricci curvature of "Markov chains" on metric spaces:

Recently, Y. Ollivier gave a new definition of the Ricci curvature, which extended all the above results for positive Ricci curvature for metric measure spaces, and can be easy to check.

Suppose the Ricci curvature $\geqslant K > 0$, then he proved that, a random walk (Markov chain) on metric spaces has the following properties:

Existence and Uniqueness of invariant measure;

L^1 and L^2-version of Bonnet-Myers theorem;

Poincaré inequality;

Gaussian Concentration (Lévy-Gromov);

Modified Log-Sobolev inequality

Gromov-Hausdorff continuity of Ricci Curvatures on a sequence of metric spaces.

In particular, he gave a open problem (i.e. problem 14) in the Lecture in Toulouse that the Log-Sobolev inequality not a modified Log-Sobolev inequality should be proved.

So, the aim of this talk is to prove a standard Log-Sobolev inequality and to consider its applications to Fractal Geometry.

2. Definitions and Notations:

- "Markov chain" on metric spaces:

Let (X, d) be a Polish metric space, a family of probability measures $m \equiv (m_x : x \in X)$ on X is called a "random walk" or "Markov chain" on X if

(i) $x \in X \longrightarrow m_x$ is measurable;

(ii) $\exists\, o \in X$ s.t. $\int d(o, y)\, m_x(dy) < +\infty$, $(\forall x \in X)$.

- Wasserstein transportation distance:

Let ν_1, ν_2 be two probability measures on X, $\Pi(\nu_1, \nu_2)$ be the set of all coupling measures of ν_1 and ν_2, and then the Wasserstein transportation distance between ν_1 and ν_2 is defined by

$$\mathcal{T}_1(\nu_1, \nu_2) = \inf_{\nu \in \Pi(\nu_1, \nu_2)} \int_{X \times X} d(x, y)\, \nu(dx\, dy).$$

- Ricci curvature of $m = (m_x : x \in X)$:
  The Ricci curvature of (X, d, m) along (xy), $x, y \in X$, is defined by
  $$k(x,y) = 1 - \frac{\mathcal{T}_1(m_x, m_y)}{d(x,y)}.$$

Remarks:

- $m = (m_x : x \in X)$ is just a probabilistic transition kernel for a Markov chain, and $k(x,y)$ is just the coefficients of contraction of the Markov chain in transportation distance, it is widely used by Dobrushin, M.F. chen, ect., especially appeared in "Dobrushin criterion" for spin systems.
- In speeial cases, some results about speetral gap (e.g. M.F. Chen, F.Y. Wang), Concentration of measures (e.g. K. Marton,

Djellout – Guillin – L.M. Wu) were proved.

- There are a lot of examples about random walk m with positive curvature, for example:

① Metric measure space (X, d, μ) with positive Ricci curvature, $\mathrm{supp}(\mu) = X$,

$$m_X = \mu|_{B(x,\varepsilon)} \Big/ \mu(B(x,\varepsilon)), \quad \varepsilon > 0.$$

In particular, X is a Riemannian manifold with positive Ricci curvature.

② Simple random walk on $\mathbb{Z}^N$, where $k(x,y) = 0$.

③ Discrete O–U process on $X = \{-N, \cdots, N\}$, where $x \sim y \Rightarrow k(x,y) = \frac{1}{2N}$.

④ Diffusion process (X_t) on Riemannian manifold with

$$dX_t = F\,dt + dB_t,$$

where, Bakry-Emery curvature for $L = \frac{1}{2}\Delta + F.\nabla$ is $\frac{1}{2} Ric_M - \nabla^{sym} F$, and when $d(x,y)$ is small enough, then $k(x,y)$ is determined by $\frac{1}{2} Ric_M - \nabla^{sym} F$.

5. Kac's random walk on $SO(N)$, where $k(x,y) \sim \frac{1}{N^2}$.

6. Glauber Dynamics for the Ising model over a finite graph.

7. $M/M/\infty$ queues.

Notations:

Let (X, d) be a Polish metric space, and $m = (m_x : x \in X)$ be a random walk,

- $\sigma_\infty(x) = \frac{1}{2} \operatorname{diam} \operatorname{Supp}(m_x)$,

  $\sigma_\infty = \sup_{x \in X} \sigma_\infty(x)$;

- λ-range gradient of $f: X \to \mathbb{R}$: $(\lambda > 0)$

$$(Df)(x) \triangleq (D_\lambda f)(x) = \sup_{y', y \in X} \frac{|f(y) - f(y')|}{d(y, y')} e^{-\lambda d(x,y) - \lambda d(x,y')};$$

If f is smooth and X is a compact Riemannian manifold, then $(D_\lambda f)(x) \longrightarrow |\nabla f(x)|$ as $\lambda \to +\infty$.

- averaging operator:

$$Mf(x) = \int f(y)\, m_x(dy).$$

- Variance and entropy of $f: X \to \mathbb{R}$:

Let ν be any probability measure on X, then

$$\mathrm{Var}_\nu(f) = \frac{1}{2} \int |f(x) - f(y)|^2 \nu(dx)\nu(dy),$$

$$\mathrm{Ent}_\nu(f) = \int_X f \log \frac{f}{\|f\|_{L^1(\nu)}} d\nu, \quad f \geq 0.$$

- Unstablity: $\bar{U}(x,y) = k_-(x,y)/k(x,y)$,

$$\bar{U} = \sup_{x \neq y} \bar{U}(x,y).$$

3. Y. Ollivier's Result on LSI:

In his paper, Y. Ollivier proved the following modified LSI:

Let $m = (m_x : x \in X)$ be a random walk on (X, d), $\sigma_\infty < +\infty$ and $k(x,y) \geq k > 0$. Then for small enough $\lambda > 0$ we have: $\forall f \geq 0$,

$$\mathrm{Ent}_\mu(f) \leq \frac{4}{k} C_m \int \frac{(Df)^2}{f} \, d\mu,$$

where μ is the invariant measure of m, and

$$C_m = \sup_{x \in X} \left\{ \mathrm{Var}_{m_x}(f) : f \text{ is 1-Lipschitz} \right\}.$$

In his proof, he proved or used following important results:

A. Associativity of entropy:

$$\mathrm{Ent}_\mu f = \sum_{t \in \mathbb{Z}_+} \int_X \mathrm{Ent}_{m_x}(M^t f)\, \mu(dx),$$

(e.g. See Th. D.13 in Dembo - Zeitouni's book on Large deviations);

$$\mathrm{Ent}_{m_x} f \le \frac{1}{Mf(x)} \mathrm{Var}_{m_x} f,$$

(using $a \log a \le a^2 - a$).

B. Gradient Contraction:

for λ small enough, $Df < +\infty$,

$$(DMf)(x) \le \left(1 - \frac{k}{2}\right) M(Df)(x).$$

C. If $Df < +\infty$, then $\forall x \in X$

$$|f(y) - f(z)| \le d(y,z)\, e^{4\lambda\sigma_\infty} M(Df)(x),$$

$\forall\, y, z \in \mathrm{Supp}(m_x)$.

4. Standard LSI :

Using the facts A, B, C in the above and the following Lemma that we proved

Lemma D: Let $D_\lambda f < +\infty$, then $\forall x \in X$ we have:

$$D_{3\lambda} f^2(x) \leq 2 f(x) D_\lambda f(x) + \frac{2}{e\lambda} (D_\lambda f)^2(x);$$

we proved the following standard Log-Sobolev inequality:

Theorem A: Let $m = (m_x : x \in X)$ be a random walk on (X, d), $\sigma_\infty < +\infty$, $k(x,y) \geq k > 0$ $(\forall x, y \in X)$. Then for small enough $\lambda > 0$, there is a universal constant $N > 0$

such that: $\forall f \in L^2(\mu)$,

$$\mathrm{Ent}_\mu(f^2) \le \frac{N\sigma_\infty^2(1+\bar{v})}{k}\int (Df)^2 d\mu$$

Moreover, for any $\xi \in [1,2]$ there is a positive constant $C(\xi, k) > 0$ such that: $\forall f \in L^1(\mu)$ and $f \geq 0$ μ-a.e.,

$$\mathrm{Ent}_\mu(f) \le C(\xi, k)\int \left|\frac{Df}{f}\right|^\xi f\, d\mu .$$

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Remark: In fact, we can prove that:

$$N \le 2^5 \times 3^2 \times 5^2 \times e^{-\frac{14}{15}},$$

$$C(\xi, k) \le \frac{\sqrt{2}}{k}\sup_{x\in X}\left(\int_{X\times X} d(y,z)^{\frac{\xi}{\xi-1}} m_x(dy) m_x(dz)\right)^{\xi-1}.$$

The above results are the improving for corresponding results in Y. Ollivier's paper.

We also improve Y. Ollivier's Gaussian concentration result by avoided the condition $\sigma_\infty < +\infty$:

Proposition: Let $\varphi : X \to \mathbb{R}$ be a α-Lipschitz function with $\alpha \le 1$. Assume $\sup_x \frac{\sigma(x)^2}{n_x} \le B^2$ and $0 \le \lambda \le \frac{1}{2B}$. Then for $x \in X$

$$(Me^{\lambda\varphi})(x) \le e^{\lambda M\varphi(x) + 4\lambda^2\alpha^2\frac{\sigma(x)^2}{n_x}}.$$

Moreover, if $x \longmapsto \frac{\sigma(x)^2}{n_x k}$ is C-Lipschitz, then for $0 \le \lambda \le \frac{1}{2B} \wedge \frac{1}{8C}$,

$$\mathbb{E}_\mu e^{\lambda(f - \mathbb{E}_\mu f)} \le e^{\frac{16}{3}\lambda^2 \mathbb{E}_\mu \frac{\sigma(x)^2}{n_x k}}$$

for any 1-Lipschitz function f. #

5. Applications for Fractal Geometry:

Let μ be a Borel probability measure on (X,d), and $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(\mu)$. Define

$$R(x,y) = \sup_{\mathcal{E}(f,f)\neq 0} \frac{|f(x)-f(y)|^2}{\mathcal{E}(f,f)}, \quad x,y \in X.$$

$R(\cdot,\cdot)$ is so called "effective resistance" of $(\mathcal{E},\mathcal{F})$.

We assume that:

- $R(x,y) < +\infty$, $\forall x, y \in X$;
- $R(\cdot,\cdot)^{1/2}$ is a distance on X, and it induces the same topology as d.

In this case, we called $(\mathcal{E},\mathcal{F})$ is a resistance form. In fractal geometry there are many Dirichlet forms (e.g. Dirichlet form associated with Brownian motions on many fractal sets) are resistance forms.

By the results in [Gong-Wu 06] we know that, spectral gap property of $(\mathcal{E}, \mathcal{F})$ usually concerns with the resolvents of $(\mathcal{E}, \mathcal{F})$. Hence, we define the following measure kernels $m = (m_x : x \in X)$ on $(X, R^{\frac{1}{2}})$: For $\alpha > 0$,

$$m_x = \alpha G_\alpha(x, \cdot) = \alpha \int_0^\infty e^{-\alpha t} p_t(x, \cdot)\, dt, \quad \forall x \in X.$$

Then $m = (m_x : x \in X)$ is a random walk on $(X, R^{\frac{1}{2}})$ if $(\mathcal{E}, \mathcal{F})$ is conservative.

We can easily prove that

Theorem B: $(X, R^{\frac{1}{2}}, m)$ has positive Ricci curvature if $2\alpha \int R(0, x) \mu(dx) < 1$.

1). If $\sup_{x,y} R^2(x, y) = D < +\infty$, then the Markov process (X_t) associated with $(\mathcal{E}, \mathcal{F})$ has an unique invariant measure μ, and the following log-Sobolev inequality holds: $\forall f \in \mathcal{F}$

$$\mathrm{Ent}_\mu(f^2) \le 2D^2 \mathcal{E}(f,f).$$

2). If $\int R(0,x)\mu(dx) < +\infty$, then the poincaré inequality holds: $\forall f \in \mathcal{F}$

$$\mathrm{Var}_\mu(f) \le C_p \mathcal{E}(f,f)$$

with $C_p \le \int R(0,x)\mu(dx)$. Moreove, we have the following Gaussian concentration: for $f \in \mathcal{F}$ with $\mathcal{E}(f,f) \le 1$

$$\mathbb{E}_\mu e^{\lambda(f - \mathbb{E}_\mu f)} \le e^{4C_p\lambda^2}$$

for any $0 \le \lambda \le \frac{1}{2\sqrt{C_p}}$.

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⚠ For Theorem B 2) we have the following example: $X = \mathbb{R}$, $\mu(dx) = e^{-cx^\alpha}\,dx$ for $\alpha > 2$, $\mathcal{E}(f,f) = \int |f'|^2 d\mu$. Then

$$\int R(0,x)\mu(dx) < +\infty, \quad \sup_{x,y} R(x,y) = +\infty.$$

⚠ For Theorem B 1) we have the following important examples in fractal geometry:

⚠) Let (X, d) be the Sierpinski gasket, μ be the normalized Hausdorff measure on X, and $(\mathcal{E}, \mathcal{F})$ be given by Example 3.1.5 in J. Kigami's book "Analysis on fractals" (Cambridge, 2001), then Theorem B 1) holds with $D^2 \leq \frac{6}{5}$.

Similar results hold for other post-critical-finite sets with regular harmonic structures.

△). Let (X,d) be the Sierpinski carpet, μ be the normalized Hausdorff measure on X, and $(\mathcal{E},\mathcal{F})$ be the self-similar Dirichlet form defined by S. Kusuoka and X.Y. Zhou in 1992, PTRF, then Theorem B 1) holds.

△). Let X be a self-similar set in a locally compact Polish space (K,d_0) with $d_f<2$, which satisfies open set condition and admits a geodesic metric $d_r \sim d_0^{\theta}$ for some $\theta>0$, and μ be the normalized Hausdorff measure on X. Then by the results in K.T. Sturm's paper on Ann. Prob. 1998 we can construct a Dirichlet form $(\mathcal{E},\mathcal{F})$ in $L^2(\mu)$. For this $(\mathcal{E},\mathcal{F})$, Theorem B 1) also holds. #

Note: The abstract of Y. Ollivier's paper has published on C. R. Math. Acad. Sci. Paris 345 (2007) n°11.

Thank you!