Functional large deviations and moderate deviations for Markov modulated risk models with reinsurance

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Outline

- Markov modulated risk model. We introduce definition of Markov modulated risk model.

- Exponential martingale and Laplace functional. We construct an exponential martingale associated with the Markov modulated risk model which plays an important role.

- Main results. The functional LDP and MDP are established in this part.

- Other models. The exponential martingale method is applied to delayed claim risk models and risk processes with Poisson shot noise Cox process.

1. Markov modulated risk model

Let $J = \{J(t), t \geq 0\}$ be an irreducible continuous time Markov chain with finite state space E and let $\pi_i, i \in E$ denote the stationary distribution of the Markov chain J.

Let $\{U_l, l \geq 1\}$ be a sequence of positive random variables and let $G_i,\ i \in E$ be probability distributions with supports in $[0, +\infty)$. Assume that for all $i \in E$, $\mu_i := \int_0^\infty x G_i(dx) < \infty$ and that $\lambda_i,\ i \in E$ are positive numbers,

Let $N(t) = \sum_{l=1}^\infty I_{\{T_l \leq t\}}$ be a Markov modulated Poisson process where $T_1 > 0\ \ a.s.$, $T_l < T_{l+1}$ on $\{T_l < \infty\}$ and $\lim_{l\to\infty} T_l = \infty$, i.e.,

$$\mathbb{E}(\exp\{i\theta(N(t) - N(s))\}|\mathcal{F}_s) = \exp\left\{(e^{i\theta} - 1)\int_s^t \lambda_{J(u)} du\right\}. \qquad (1)$$

where $\mathcal{F}_s = \sigma(N(u), u \leq s) \vee \sigma(J(u), u \geq 0)$.

A reinsurance policy is a measurable function from $\mathcal{R} : [0\,,\infty)\times[0\,,\infty) \to [0\,,\infty)$ which satisfies $0 \le R_t(\alpha) \le \alpha$, where $R_t(\alpha) = \mathcal{R}(t\,,\alpha)$.

A Markov modulated risk process with a reinsurance policy $\mathcal{R}$ is defined by

$$X^x_{\mathcal{R}}(t) = x + p_{\mathcal{R}}(t) - S_{\mathcal{R}}(t), \qquad (2)$$

where $x > 0$ is the initial capital; $p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t)$ is the deterministic premium, $p = (1+\kappa)\sum_{i\in E}\pi_i\lambda_i\mu_i$ is constant premium rate, $q_{\mathcal{R}}(t) = (1+\eta)\sum_{i\in E}\pi_i\lambda_i\int_0^t(\mu_i - \int_0^\infty R_s(x)G_i(dx))\,ds$ is the premium up to time t paid by the insurer to reinsurer;

$$S_{\mathcal{R}}(t) = \sum_{l=1}^{N(t)} R_{T_l}(U_l) \qquad (3)$$

is aggregate claim process.

The Markov modulated risk process with reinsurance is a generalization of classical case. For examples, if $G_i = G$ and $\lambda_i = \lambda$ for all $i \in E$, and $\{U_l, l \geq 1\}$, $\{N(t), t \geq 0\}$ and J are independent, then $S_{\mathcal{R}}(t)$ is the classic case. Recently, Macci and Stabile *JAP 43(2006),* 713–728) studied the large deviations and the ruin probability of risk processes with reinsurance and obtained a functional large deviation principle for the classic case. In this talk, we present an exponential martingale method to establish large deviations and moderate deviations for risk processes, including Markov modulated risk models, delayed claim risk models, risk processes with the Poisson shot noise Cox process, etc.

2. Exponential martingale and Laplace functional

Lemma 1 *(1). Set*

$$M_t := S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty R_u(x) G_{J(u)}(dx) \lambda_{J(u)} du. \tag{4}$$

Then $\{M_t, \mathcal{G}_t, t \geq 0\}$ *is a martingale where* $\mathcal{G}_s = \sigma(N(u), u \leq s) \vee \sigma(J(u), u \geq 0) \vee \sigma(U_l, l \leq N(s))$.

(2). If for some $\delta > 0$,

$$\sup_{i \in E} \int_0^\infty e^{\delta x} G_i(dx) < \infty,$$

then for any measurable function $\theta(t)$ *satisfying* $\sup_{t \geq 0} \theta(t) < \delta$,

$$Z_t^\theta := \exp\left\{ \int_0^t \theta(u) dS_{\mathcal{R}}(u) - \int_0^t \int_0^\infty \left(e^{\theta(u) R_u(x)} - 1 \right) G_{J(u)}(dx) \lambda_{J(u)} du \right\} \tag{5}$$

is a $\{\mathcal{G}_t\}$*-martingale.*

Proof of (2). Set $L_t = \int_0^t \theta(u) dS_{\mathcal{R}}(u)$. Applying Itô formula to e^{L_t}, we have

$$
\begin{aligned}
e^{L_t} =& 1 + \int_0^t e^{L_{u-}} \theta(u) dS_{\mathcal{R}}(u) + \sum_{0<u\le t} e^{L_{u-}} \left(e^{\triangle L_u} - 1 - \triangle L_u \right) \\
=& 1 + \int_0^t e^{L_{u-}} \theta(u) dM_u + \int_0^t \int_0^\infty e^{L_{u-}} \theta(u) R_u(x) G_{J(u)}(dx) \lambda_{J(u)} du \\
& + \sum_{l=1}^{\infty} e^{L_{T_{l-1}}} \left(e^{\theta(T_l) R_{T_l}(U_l)} - 1 - \theta(T_l) R_{T_l}(U_l) \right) I_{\{T_l \le t\}}
\end{aligned}
$$

where $\triangle L_u = L_u - L_{u-}$. Conditioning expectation both sides the above equation on $\sigma(J(s), s \ge 0)$, we can get

$$
\mathbb{E}\left(e^{L_t} | J\right) = 1 + \int_0^t \int_0^\infty \mathbb{E}\left(e^{L_{u-}} | J\right) \left(e^{\theta(u) R_u(x)} - 1\right) G_{J(u)}(dx) \lambda_{J(u)} du
$$

which implies

$$
\mathbb{E}\left(e^{L_t} | J\right) = \exp\left\{ \int_0^t \int_0^\infty \left(e^{\theta(u) R_u(x)} - 1\right) G_{J(u)}(dx) \lambda_{J(u)} du \right\}.
$$

Corollary 1 *If for some* $\delta > 0$,

$$\sup_{i \in E} \int_0^\infty e^{\delta x} G_i(dx) < \infty,$$

then for any $m \geq 1$, $0 = t_0 < t_1 < \cdots < t_m$ *and* $\theta_1, \cdots, \theta_m \in (-\infty, \delta)$,

$$\begin{aligned} &\prod_{l=1}^{m} \inf_{i \in E} \mathbb{E}_i \left(\exp \left\{ \int_0^{t_l - t_{l-1}} V_l(u, J(u)) du \right\} \right) \\ \leq &\mathbb{E} \left(\exp \left\{ \sum_{l=1}^{m} \theta_l \sum_{n=N(t_{l-1})+1}^{N(t_l)} R_{T_n}(U_n) \right\} \right) \\ \leq &\prod_{l=1}^{m} \sup_{i \in E} \mathbb{E}_i \left(\exp \left\{ \int_0^{t_l - t_{l-1}} V_l(u, J(u)) du \right\} \right) \end{aligned} \tag{6}$$

where $\mathbb{E}_i(\cdot) := \mathbb{E}(\cdot | J(0) = i)$ *and*

$$V_l(u, z) = \int_0^\infty \left(e^{\theta_l R_{u + t_{l-1}}(x)} - 1 \right) \lambda_z G_z(dx).$$

3. Main Results

3.1. Large deviations

Assumptions:

$(H1)$. There exists a measurable function $\tilde{R} : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t \to \infty} \sup_{x \in [0,\infty)} |R_t(x) - \tilde{R}(x)|/(x+1) = 0.$$

$(H2)$. For all $r > 0$,

$$\sup_{i \in E} \int_0^\infty e^{rx} G_i(dx) < \infty.$$

Let $Q = (q_{ij})_{i,j \in E}$ be the intensity matrix of the Markov chain $\{J(t), t \geq 0\}$. For any vector $\mathbf{v} = (v_i)_{i \in E}$, set $Q(\mathbf{v}) = (q_{ij} + \delta_{ij} v_i)_{i,j \in E}$ and let

$\Lambda(\mathbf{v})$ be the logarithm of the simple and positive eigenvalue of the exponential matrix $e^{Q(\mathbf{v})}$. By the LDP for the Markov chain $\{J(t), t \geq 0\}$ (cf. Donsker-Varadhan(1976, CPAM),Baldi-Piccioni (1999, SPL), Wu(2000,JFA)), for any $j \in E$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_j \left(\exp \left\{ \int_0^t v_{J(u)} du \right\} \right) = \Lambda(\mathbf{v}). \tag{7}$$

Therefore, under assumptions (H1) and (H2), (7) implies that for any $j \in E$ and for any $\theta \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_j \left(\exp \left(\theta S_{\mathcal{R}}(t) \right) \right) = \Lambda \left(\left(\lambda_i \int_0^\infty (e^{\theta \tilde{R}(x)} - 1) G_i(dx) \right)_{i \in E} \right). \tag{8}$$

Define

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} \left\{ \theta x - \Lambda \left(\left(\lambda_i \int_0^\infty (e^{\theta \tilde{R}(z)} - 1) G_i(dz) \right)_{i \in E} \right) \right\}. \tag{9}$$

Theorem 1 *Let assumptions (H1) and (H2) hold. Then*

$$\left\{\mathbb{P}\left(\left(\frac{S_{\mathcal{R}}(\alpha t)}{\alpha}\right)_{t\in[0,1]} \in \cdot\right), \alpha > 0\right\}$$

satisfies the LDP in $D[0,1]$ *with speed* α *and rate function* $I^{(ld)}$ *defined by*

$$I^{(ld)}(f) = \begin{cases} \int_0^1 \Lambda^*(\dot{f}(t))dt & \text{if } f(0) = 0 \text{ and } f \text{ is absolutely continuous} \\ +\infty & \text{otherwise} \end{cases} \tag{10}$$

i.e., for any closed set F *and open set* G *in* $D([0,1])$ *,*

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha} \log \mathbb{P}\left(\frac{S_{\mathcal{R}}(\alpha\cdot)}{\alpha} \in F\right) \leq -\inf_{f\in F} I^{(ld)}(f),$$

$$\liminf_{\alpha\to\infty} \frac{1}{\alpha} \log \mathbb{P}\left(\frac{S_{\mathcal{R}}(\alpha\cdot)}{\alpha} \in G\right) \geq -\inf_{f\in G} I^{(ld)}(f).$$

Lemma 2 *Let assumptions (H1) and (H2) hold. Then for any* $m \geq 1$*,* *and* $0 = t_0 \leq t_1 < t_2 < \cdots < t_m \leq 1$,

$$\left\{ \left(\frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_1), \frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_2), \cdots, \frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_m) \right), \alpha > 0 \right\}$$

satisfies the LDP with speed α *and with rate function* $I^{(ld)}_{t_1,\ldots,t_m}$ *defined by*

$$I^{(ld)}_{t_1,\ldots,t_m}(x_1, \cdots, x_m) = \sum_{l=1}^{m} (t_l - t_{l-1}) \Lambda^* \left(\frac{x_l - x_{l-1}}{t_l - t_{l-1}} \right) \tag{11}$$

where $x_0 = 0$.

Lemma 3 *for any* $t \in [0,1]$ *and for any* $\eta > 0$,

$$\lim_{\delta \downarrow 0} \lim_{\alpha \to \infty} \frac{1}{\alpha} \log \mathbb{P} \left(\frac{1}{\alpha} \sup_{t \leq s \leq t+\delta} |S_{\mathcal{R}}(\alpha t) - S_{\mathcal{R}}(\alpha s)| \geq \eta \right) = -\infty. \tag{12}$$

Proof By Lemma 1, for any $\beta \in \mathbb{R}$, $(Z^{\beta}_t)^{-1} Z^{\beta}_{t+s}, s \geq 0$ is a martingale under probability $\mathbb{P}(\cdot|J)$, where

$$Z^{\beta}_t := \exp \left\{ \beta S_{\mathcal{R}}(t) - \int_0^t \int_0^{\infty} \left(e^{\beta R_u(x)} - 1 \right) G_{J(u)}(dx) \lambda_{J(u)} du \right\}.$$

Then by the maximum inequality for martingale, we have that for any $\beta > 0$

$$
\begin{aligned}
&\frac{1}{\alpha}\log\mathbb{P}\left(\frac{1}{\alpha}\sup_{t\le s\le t+\delta}|S_{\mathcal{R}}(\alpha t)-S_{\mathcal{R}}(\alpha s)|\ge\eta\right)\\
=&\frac{1}{\alpha}\log\mathbb{P}\left(\frac{1}{\alpha}\sup_{t\le s\le t+\delta}(S_{\mathcal{R}}(\alpha t)-S_{\mathcal{R}}(\alpha s))\ge\eta\right)\\
\le&\frac{1}{\alpha}\log\mathbb{E}\left(\mathbb{P}\left(\sup_{0\le s\le\delta}(Z^{\beta}_{\alpha t})^{-1}Z^{\beta}_{\alpha(t+s)}\ge e^{\alpha\beta\eta-\alpha\delta C(\beta)}\middle| J\right)\right)\\
\le&\frac{1}{\alpha}\log\mathbb{E}\left(e^{-\alpha\beta\eta+\alpha\delta C(\beta)}\mathbb{E}\left((Z^{\beta}_{\alpha t})^{-1}Z^{\beta}_{\alpha(t+\delta)}\middle| J\right)\right)=-\beta\eta+\delta C(\beta)
\end{aligned}
$$

where

$$C(\beta):=\sup_{i\in E}\lambda_i\int_0^{\infty}\left(e^{\beta x}-1\right)G_i(dx).$$

Now letting $\alpha\to\infty$ firstly, then $\delta\downarrow 0$, and $\beta\to\infty$ finally, we get (12).

3.2. Moderate deviations

$\{a(t), t \geq 0\}$ denotes a positive function satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{a(t)}{\sqrt{t}} = \infty. \tag{13}$$

We introduce the following assumptions:

$(H3)$. There exist two non-negative measurable functions $\widehat{R}(x)$ and $m(u)$ such that for all $u \geq 0$ and $x > 0$, $|R_u(x) - \widehat{R}(x)| \leq m(u)(x+1)$ and

$$\lim_{u\to\infty} m(u) = 0, \qquad \lim_{t\to\infty} \frac{1}{a(t)} \int_0^t m(u)du = 0.$$

$(H4)$. There exists $\delta > 0$ such that

$$\sup_{i\in E} \int_0^\infty e^{\delta x} G_i(dx) < \infty.$$

For example, if $R_t(x) = c(1-1/(1+t)^\gamma)x^\tau$ where $c, \tau \in (0,1]$ and $\gamma > 0$, then $(H3)$ holds for $\widehat{R}(x) = cx^\tau$, $m(u) = 1/(1+u)^\gamma$ and $a(t) = t^\beta$ where $\max\{1-\gamma, 1/2\} < \beta < 1$.

Let $P(t) = (p_{ij}(t))_{i,j\in E} = e^{tQ}$ be the semigroup of the Markov chain J. Since $\{J(t), t \geq 0\}$ is uniformly ergodic, the following conclusions are known.

(1). There exists $c > 0$ such that for any function f on E

$$\sup_{i\in E}\left|\sum_{k\in E} p_{ik}(t)f(k) - \sum_{j\in E}\pi_j f(j)\right| \leq e^{-ct}\sup_{i\in E}|f(i)|.$$

(2). For any $j \in E$ and any function f on E,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_j\left(\int_0^t (f(J(u)) - \mathbb{E}_\pi(f(J(u)))du\right)^2\\ =&2\int_0^\infty \sum_{i\in E}\pi_i\left(f(i) - \sum_{k\in E}\pi_k f(k)\right)\sum_{k\in E}p_{ik}(u)f(k)du.\end{aligned}$$

Set $\widehat{R}_i = \int_0^\infty \widehat{R}(x) G_i(dx)$. Then

$$\sigma_1^2 := \lim_{t\to\infty} \frac{1}{t} \mathbb{E}_\pi \left(\left(\int_0^t \left(\widehat{R}_{J(u)} \lambda_{J(u)} - \mathbb{E}_\pi \left(\widehat{R}_{J(u)} \lambda_{J(u)} \right) \right) du \right)^2 \right) \tag{14}$$

exists and

$$\sigma_1^2 = 2 \int_0^\infty \sum_{j\in E} \pi_j \left(\lambda_j \widehat{R}_j - \sum_{i\in E} \pi_i \lambda_i \widehat{R}_i \right) \sum_{k\in E} p_{jk}(t) \lambda_k \widehat{R}_k dt. \tag{15}$$

By the MDP for the Markov chain $\{J(t), t \geq 0\}$ (cf. Wu(1995, AP), Gao(1995, Acta Mathematica Scientia)), for any $i \in E$ and any function f on E,

$$\begin{aligned} &\lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log E_i \left(\exp \left\{ \frac{a(\alpha)}{\alpha} \int_0^{\alpha t} \Big(f(J(u)) - \mathbb{E}_\pi(f(J(u))) \Big) du \right\} \right) \\ =& t \int_0^\infty \sum_{i\in E} \pi_i \left(f(i) - \sum_{k\in E} \pi_k f(k) \right) \sum_{k\in E} p_{ik}(u) f(k) du. \end{aligned}$$

Theorem 2 *Let assumption (H3) and (H4) hold. Then*

$$\left\{\mathbb{P}\Big(\frac{S_{\mathcal{R}}(\alpha t)-\alpha t\sum_{i\in E}\pi_i\lambda_i\hat{R}_i}{a(\alpha)}\Big|_{t\in[0,1]}\in\ \cdot\ \Big),\alpha>0\right\}$$

satisfies the LDP in $D[0,1]$ with speed $\frac{a^2(\alpha)}{\alpha}$ and rate function $I^{(md)}$ defined by

$$I^{(md)}(f)=\begin{cases}\frac{1}{2\sigma^2}\int_0^1(\dot{f}(t))^2dt & \text{if } f(0)=0 \text{ and } f \text{ is absolutely continuous}\\ +\infty & \text{otherwise.}\end{cases} \tag{16}$$

Lemma 4 *Let assumptions (H3) and (H4) hold. Set $\bar{S}_{\mathcal{R}}(\cdot)=S_{\mathcal{R}}(\cdot)-\mathbb{E}_\pi(S_{\mathcal{R}}(\cdot))$. Then for any $j\in E$, $\theta\in\mathbb{R}$ and $t>0$*

$$\lim_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\mathbb{E}_j\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right)=\frac{1}{2}\theta^2\sigma^2 t. \tag{17}$$

where $\sigma^2=\sigma_1^2+\sigma_2^2$, and

$$\sigma_2^2=\sum_{i\in E}\left(\pi_i\lambda_i\int_0^\infty\hat{R}^2(x)G_i(dx)\right).$$

Lemma 5 *for any $t \in [0,1]$ and for any $\eta > 0$,*

$$\lim_{\delta\downarrow 0}\lim_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\mathbb{P}\left(\frac{1}{a(\alpha)}\sup_{t\le s\le t+\delta}|\bar{S}_{\mathcal{R}}(\alpha t)-\bar{S}_{\mathcal{R}}(\alpha s)|\ge\eta\right)=-\infty. \quad (18)$$

Proof For any $\beta > 0$

$$\begin{aligned}
&\frac{\alpha}{a^2(\alpha)}\log\mathbb{P}\left(\frac{1}{a(\alpha)}\sup_{t\le s\le t+\delta}(\bar{S}_{\mathcal{R}}(\alpha t)-\bar{S}_{\mathcal{R}}(\alpha s))\ge\eta\right)\\
\le&\frac{\alpha}{a^2(\alpha)}\log\mathbb{E}\left(e^{-\frac{a^2(\alpha)\beta\eta}{\alpha}+\alpha\delta C(\alpha,\beta)}\mathbb{E}\left((Z_{\alpha t}^{\beta})^{-1}Z_{\alpha(t+\delta)}^{\beta}\Big|J\right)\right) \quad (19)\\
=&-\beta\eta+\frac{\alpha^2}{a^2(\alpha)}\delta C(\alpha,\beta)
\end{aligned}$$

where

$$C(\alpha,\beta):=\sup_{i\in E}\lambda_i\int_0^\infty\left(e^{a(\alpha)\beta x/\alpha}-1-a(\alpha)\beta x/\alpha\right)G_i(dx)=O(a^2(\alpha)/\alpha^2).$$

Now letting $\alpha\to\infty$ firstly, then $\delta\downarrow 0$, and $\beta\to\infty$ finally, we get that

$$\lim_{\delta\downarrow 0}\lim_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\mathbb{P}\left(\frac{1}{a(\alpha)}\sup_{t\le s\le t+\delta}(\bar{S}_{\mathcal{R}}(\alpha t)-\bar{S}_{\mathcal{R}}(\alpha s))\ge\eta\right)=-\infty.$$

3.3. An estimate for the ruin probability

The ruin time and the ruin probability are defined by

$$\tau_x = \inf\{t \geq 0; X_{\mathcal{R}}^x(t) < 0\}, \qquad \psi(x) = \mathbb{P}(\tau_x < \infty). \tag{20}$$

Theorem 3 *Let assumptions (H1) and (H2) hold. Set*

$$R := \sup\left\{ r > 0;\ \inf_{t \geq 0} \left(r p_{\mathcal{R}}(t) - t \sup_{i \in E} \lambda_i \int_0^\infty (e^{rx} - 1)\, G_i(dx) \right) \geq 0 \right\}. \tag{21}$$

Then

$$\psi(x) \leq e^{-Rx}. \tag{22}$$

Proof Without loss of generality we assume $0 < R < \infty$. By Theorem 1, for any $\beta \in \mathbb{R}$,

$$Z_t^\beta := \exp\left\{ \beta S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty \left(e^{\beta R_u(x)} - 1 \right) G_{J(u)}(dx) \lambda_{J(u)} du \right\}.$$

is a martingale under probability $\mathbb{P}(\cdot|J)$. Therefore, by Doob stopping time theorem, we have that for any $\beta > 0$ and any $t \in [0, \infty)$, $\mathbb{E}\left(Z^{\beta}_{\tau_x \wedge t}\right) = 1$ which implies that $\mathbb{E}\left(Z^{\beta}_{\tau_x} I_{\{\tau_x < \infty\}}\right) = 1$. Therefore

$$\begin{aligned}
&\psi(x) = \mathbb{P}(S_{\mathcal{R}}(\tau_x) \geq x + p_{\mathcal{R}}(\tau_x), \tau_x < \infty) \\
\leq &e^{-Rx}\mathbb{E}\left(Z^{R}_{\tau_x} \exp\left\{-Rp_{\mathcal{R}}(\tau_x) + \int_0^{\tau_x}\int_0^{\infty}\left(e^{RR_u(x)} - 1\right) G_{J(u)}(dx)\lambda_{J(u)}du\right\} I_{\{\tau_x < \infty
\leq &e^{-Rx}\mathbb{E}\left(Z^{R}_{\tau_x} \exp\left\{-\left(Rp_{\mathcal{R}}(\tau_x) - \tau_x \sup_{i \in E} \lambda_i \int_0^{\infty}\left(e^{Rx} - 1\right) G_i(dx)\right)\right\} I_{\{\tau_x < \infty\}}\right) \\
\leq &e^{-Rx}\mathbb{E}\left(Z^{R}_{\tau_x} I_{\{\tau_x < \infty\}}\right) = e^{-Rx}.
\end{aligned}$$

Here we present a numerical example in which we calculate R in Theorem 3. We consider the proportional policy, i.e. $R_t(x) = b_t x$ for some $b_t \in [0, 1]$ and assume that $\lim_{t \to \infty} b_t = b_\infty \in [0, 1]$.

Example 1 *Let J be a Markov chain with two state space $E = \{1, 2\}$ with intensity matrix*

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $\lambda_1 = 1$, $\lambda_2 = 2$ *and let* G_1, G_2 *be the exponential distributions with parameters* 1 *and* 2 *respectively. Then the corresponding stationary distribution is* $(\pi_1, \pi_2) = (\frac{1}{2}, \frac{1}{2})$. *Let* $\kappa = 4$ *and* $\eta = 5$ *be the relative safety loading for the insurer and the reinsurer respectively. Finally we assume* $b_t \geq \frac{1}{2}$. *Then for any* $0 < r < 1$,

$$rp_{\mathcal{R}}(t) - t \sup_{i=1,2} \lambda_i \int_0^\infty (e^{rx} - 1)\, G_i(dx) \geq \frac{r(1-2r)t}{1-r}.$$

Therefore, $R \geq \frac{1}{2}$, *and corresponding ruin probability* $\psi(x) \leq e^{-\frac{1}{2}x}$.

4. Other models

- Delayed claims risk model:

$$Y_t = c + pt - \sum_{k=1}^{\infty} X_k I_{(0,t]}(T_k) - \sum_{k=1}^{\infty} Y_k I_{(0,t]}(T_k + W_k)$$

where $\{T_k, k \geq 1\}$ are the jump times of a Markov modulated Poisson process $\{N(t), t \geq 0\}$ with intensity $\lambda_{J(t)}$, $\{X_k, k \geq 1\}, \{Y_k, k \geq 1\}, \{W_k, k \geq 1\}$ and $\{N(t), t \geq 0\}$ are conditionally independent given J.

- Cox risk process with Poisson shot noise intensity:

$$Z_t = a + bt - \sum_{k=1}^{N_t} X_k$$

where the intensity of the point process $\{N_t, t \geq 0\}$

$$\lambda_t = \lambda + \sum_{n \in \mathbb{N}} h(t - \tau_n, Y_n),$$

the function $h(\cdot, \cdot)$ is nonnegative and $h(t, x) = 0$ for $t < 0$, $x \in \mathbb{R}$, $\tau_n, n \geq 1$ are the jump time of a Poisson process $\{\widetilde{N}_t, t \geq 0\}$ with intensity ρ, $Y_n, n \geq 1$ are positive i.i.d. random variables.

Thank you