Log-Sobolev Inequalities for Diffusion Processes with Unbounded Below Curvatures

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Contents

Framework and definition

M: connected non-compact complete Riemannian manifold.

$L = \Delta + \nabla V, \mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure for some $V \in C^2(M)$.

P_t the associated symmetric diffusion semigroup.

Log-Sobolev inequality:

(LS) $\mu(f^2 \log f^2) \leq C\mu(|\nabla f|^2), \quad \mu(f^2) = 1.$

In the present case, (LS) holds for some $C > 0$ if and only if P_t is hyperbounded (L. Gross, B. Bakry...):

(H) $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ for some $t > 0$.

Known results: curvature conditions

Bakry-Emery 84: $\mathrm{Ric} - \mathrm{Hess}_V \geq K > 0 \Rightarrow (LS)$.

M.-F. Chen-Wang 97: $\liminf_{\rho_o \to \infty} \{\mathrm{Ric} - \mathrm{Hess}_V\} > 0 \Rightarrow (LS)$. ρ_o is the Riemannian distance to a fixed point $o \in M$.

Wang 97, 01: (LS) holds if $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ for some $K \geq 0$ and $\mu(\mathrm{e}^{\lambda \rho_o^2}) < \infty$ for some $\lambda > K/2$. Consequently, if $\mathrm{Ric} - \mathrm{Hess}_V \geq 0$ then (LS) $\Leftrightarrow \mu(\mathrm{e}^{\varepsilon \rho_o^2}) < \infty$ for some $\varepsilon > 0$.

X. Chen-Wang 07: for any $K > 0$ and $\varepsilon < K/2$, there exist examples such that $\mu(\mathrm{e}^{\varepsilon \rho_o^2}) < \infty$ without (LS).

Proofs are based on the fact

$$\mathrm{Ric} - \mathrm{Hess}_V \geq -K \Leftrightarrow |\nabla P_t f| \leq \mathrm{e}^{Kt} P_t |\nabla f|.$$

Aim: investigate (LS) for unbounded below curvatures.

The roles of Ric and $-\mathrm{Hess}_V$: short distance

Is a condition on $\mathrm{Ric}-\mathrm{Hess}_V$ reasonable in the study of (LS)? Do Ric and $-\mathrm{Hess}_V$ play the same role for (LS)?

Small distance behavior: Let X_t be the L-diffusion process with $X_0 = x$. For $Z \in T_xM$ let $Z_t \in T_{X_t}M$ solve the differential equation

$$D_t Z_t := //_{0\to t}\frac{\mathrm{d}}{\mathrm{d}t}//_{t\to 0} Z_t = -(\mathrm{Ric} - \mathrm{Hess}_V)(Z_t), \quad Z_0 = Z.$$

Then (**Bismut-Elworthy-Li formula**)

$$\langle \nabla P_t f, Z\rangle = \mathbb{E}\langle \nabla f(X_t), Z_t\rangle.$$

In particular, $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ implies

$$|\nabla P_t f| \leq \mathrm{e}^{Kt} P_t |\nabla f|.$$

So, for short distance behaviors conditions on $\mathrm{Ric} - \mathrm{Hess}_V$ are reasonable.

The roles of Ric and $-\text{Hess}_V$: long distance

Long distance behaviors:

Ric $\geq -K$, $-\text{Hess}_V \geq -\delta$ for some $K, \delta \in \mathbb{R}$,

$\rho_o := \rho(o, \cdot)$.

(a) Second variational formula implies

$$L\rho_o \leq \sqrt{K(d-1)} \coth\left[\sqrt{K/(d-1)}\,\rho_o\right] + \delta\rho_o + |\nabla V|(o).$$

(b) Coupling by parallel displacement (X_t, Y_t)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(X_t, Y_t) \leq 2\sqrt{K(d-1)}\tanh\left[\frac{\rho(X_t, Y_t)}{2}\sqrt{K/(d-1)}\right] + \delta\rho(X_t, Y_t).$$

(c) Volume comparison theorem implies

$$\mu(B(o, r)) \leq c\exp[\sqrt{K/(d-1)}\, r + \delta r^2].$$

The roles of Ric and $-\mathrm{Hess}_V$

Conclusion: Conditions on $\mathrm{Ric} - \mathrm{Hess}_V$ are not reasonable for long distance behaviors.

Since the log-Sobolev inequality always hold on compact domains, it is most likely a long-distance behavior of the process. Therefore, below we suggest conditions on Ric and $-\mathrm{Hess}_V$ respectively, rather than on $\mathrm{Ric} - \mathrm{Hess}_V$ as before.

Main question: Given a lower bound of $-\mathrm{Hess}_V$, find the weakest possibility on lower bound of Ric such that (LS) holds.

Since (LS) implies $\mu(\mathrm{e}^{\varepsilon \rho_o^2}) < \infty$ for some $\varepsilon > 0$, it is reasonable to assume $-\mathrm{Hess}_V$ is positive for large ρ_o.

The main result

Let

$(\mathbf{2.1})$ $\quad -\mathrm{Hess}_V \geq \delta$ outside a compact set

and

$(\mathbf{2.2})$ $\quad \mathrm{Ric} \geq -(c + \sigma^2 \rho_o^2)$ for some constants $\delta, \sigma, c > 0$.

Theorem

(1) *If* $(\mathbf{2.1})$ *and* $(\mathbf{2.2})$ *with* $\delta > (1+\sqrt{2})\sigma\sqrt{d-1}$, *then (LS) holds.*

(2) *For any* $\sigma, \delta > 0$ *such that* $\delta \leq \sigma\sqrt{d-1}$, *there exist examples such that* **(2.1), (2.2)** *hold but (LS) does not.*

Let r_0 be the smallest positive constant such that for any M and V with μ a probability, conditions **(2.1), (2.2)** with $\delta > r_0\sigma\sqrt{d-1}$ imply $(\mathbf{H})$. We have

$$r_0 \in [1, 1+\sqrt{2}].$$

Exact value of r_0 is unknown.

Harnack inequality.

Below is a refinement from Arnaudon/Thalmaier/Wang (06) where ρ^4 is involved.

Theorem

Assume that $-\text{Hess}_V \geq \delta$ *outside a compact set and*

$$\text{Ric} \geq -(c + \sigma^2 \rho_o^2)$$

hold for some $\delta, \sigma, c > 0$ *with* $\delta > (1+\sqrt{2})\sigma\sqrt{d-1}$. *Then there exist* $C > 0$ *and* $\alpha > 1$ *such that*

$$(P_T f(y))^\alpha \leq (P_T f^\alpha(x)) \exp\left[\frac{C}{T}\rho(x,y)^2 + C(T + \rho_o(x)^2)\right]$$

holds for all $x, y \in M, T > 0$ *and nonnegative* $f \in C_b(M)$.

The proof.

By volume comparison theorem, conditions (2.1) and (2.2) imply $\mu(e^{\lambda\rho_o^2}) < \infty$ for some $\lambda > 0$. By the above Harnack inequality for $\alpha = 2$ and large enough $T > 0$, we have

$$(P_T f(y))^2 \leq P_T f^2(x) \exp\left[\frac{\lambda}{2}\rho_o(y)^2 + CT + C_1(T)\rho_o(x)^2\right]$$

for all f with $\mu(f^2) = 1$. Hence,

$$(P_T f(y))^2 \mu(\exp[-C_1(T)\rho_o^2]) \leq \exp\left[CT + \frac{\lambda}{2}\rho_o(y)^2\right].$$

Thus, $\|P_T\|_{2\to 4}^4 \leq C_2(T)\mu(\exp[\lambda\rho_o^2]) < \infty$.

A further question

In general, given a function γ (positive at infinity), one may try to find optimal lower bound of curvature such that

$$-\text{Hess}_V \geq \gamma \circ \rho_o$$

implies (LS). For stronger γ one may also derive the following stronger supercontractivity and ultracontractivity:

(S) $\|P_t\|_{2\to 4} < \infty$ for all $t > 0$.

(U) $\|P_t\|_{1\to \infty} < \infty$ for all $t > 0$.

Supercontractivity and Ultracontractivity

(Röckner/Wang 03) Assume $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ for some $K \geq 0$.

(1) P_t is Supercontractive $\Leftrightarrow \mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty$ for all $\lambda > 0$.

(2) P_t is Ultracontractive $\Leftrightarrow \|P_t\mathrm{e}^{\lambda\rho_o^2}\|_\infty < \infty$ for any $t, \lambda > 0$.

Example

Let Ric be bounded below, $V = -\rho^\delta \log^\alpha(1+\rho_o)$ *for some* $\delta, \alpha \geq 0$.

(1) *(LS)* $\Leftrightarrow \delta \geq 2$.

(2) **(S)** $\Leftrightarrow$ *either* $\delta > 2$ *or* $\delta = 2, \alpha > 0$.

(3) **(U)** $\Leftrightarrow$ *either* $\delta > 2$ *or* $\delta = 2, \alpha > 1$.

(S) and (U): unbounded below curvature

Since (**S**) or (**U**) $\Rightarrow \mu(\mathrm{e}^{\lambda\rho_o^2}) < \infty, \lambda > 0$, we assume

(**3.1**) $\quad -\mathrm{Hess}_V \geq \Phi \circ \rho_o$

for some increasing function Φ with $\Phi(r) \uparrow \infty$ as $r \uparrow \infty$.
Let Ψ be a strictly positive increasing function such that

(**3.2**) $\;\mathrm{Ric} \geq -\Psi \circ \rho_o$.

Theorem

P_t is supercontractive if

$$\liminf_{r\to\infty} \frac{\int_0^r \Phi(s)\mathrm{d}s}{\sqrt{\Psi(r+c)(d-1)}} > 1+\sqrt{2}, \quad c > 0;$$

and is ultracontractive if furthermore

$$\int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}\int_0^{\sqrt{r}} \Phi(s)\mathrm{d}s} < \infty.$$

Example

Example

Let $\theta, \sigma > 0$ *and* $\Phi(s) = \log^\theta(1+s), \Psi(s) = \sigma^2 s^2 \log^{2\theta}(s+1)$.

If $\sigma < 1/((1+\sqrt{2})\sqrt{d-1})$ *then* **(3.1), (3.2)** *imply* **(S)**.

If moreover $\theta > 1$ *then* **(U)** *holds and*

$$\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} \le \exp\left[c_1 + \exp(c_2 t^{-1/(\theta-1)})\right]$$

for some $c_1, c_2 > 0$ *and all* $t > 0$.

Thank You !