Workshop on Markov Processes and Related Topics

Anhui Normal University, July 24, 2008

Random Stretched Graphs

Dayue Chen

Peking University

Thank BNU for invitation,

Thank ANU for hospitality,

Salute to Prof. DING Wanding!

2008中国高校杰出青年科学基金获得者校友榜

哈工大20人

中南大学20人

西安交大18人

西北大17人：第20名

西北工大17人：

东南大学17人：

安徽师大16人： 王凤雨 王卫华 朱江 李亚栋 沈旭 肖杰 赵刚 徐飞 汪小全 田捷 沈月毛 陈传峰 毛兰群 邱小波 王启华 章璞

东北大学14人：上海交大14人：

Plan of the Talk

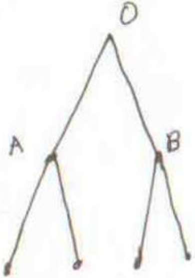
1. Definition of random stretched graph
2. anchored expansion constant
3. The critical point of percolation model
4. Contact process (I): Critical point
5. Contact process (II): Linear growth

I. Definition of a random stretched graph

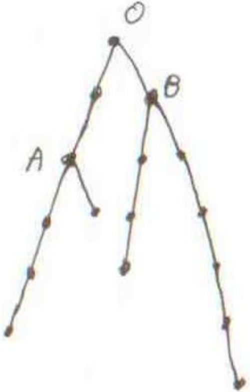
Given graph G and probability distribution ν.

Suppose that ν is concentrated on integers, and that $\{X_e, e \in E(G)\}$ are $i.i.d.$ random variables.

Replacing every edge e of G with a path of X_e edges, we obtain a random stretched graph.



The original graph G



The stretched graph $\check{G}$

Remarks:

1. Benjamini, Lyons & Schramm (1999)

2. G^ν can be a realization , or a probability measure.

3. ν can be concentrated on one point. (fixed stretch).

4. Usually the first moment of ν exists,
in some case, ν decays exponentially.

5. G itself can be a random graph.

6. If G is a G-W tree, ν is a geometric distribution, then G^ν is still a G-W tree.

II. Anchored Expansion Constant

Fix $o \in V(\mathbb{G})$, Define $\iota_E^*(\mathbb{G})$ to be

$$\lim_{n\to\infty} \inf \left\{ \frac{|\partial S|}{|S|} : o \in S \subset V(\mathbb{G}),\ S \text{ is connected},\ n \le |S| < \infty \right\}.$$

$\iota_E^*(\mathbb{G})$ is independent of the choice of o.

Very similar with the isoperimetric constant.

$$\iota_E(\mathbb{G}) := \inf \left\{ \frac{|\partial S|}{|S|} : S \subset V(\mathbb{G}),\ S \text{ is connected},\ 1 \le |S| < \infty \right\}$$

$\iota_E(\mathbb{G}) \le \iota_E^*(\mathbb{G})$.

$\iota_E^*(\mathbb{G}) > 0$ implies that the speed of SRW on G is positive.

If the support of ν is not finite, then $\iota_E(\mathbb{G}^\nu) = 0$.

But $\iota_E^*(\cdot)$ is more robust. $\iota_E^*(\mathbb{G}) > 0 \Longrightarrow \iota_E^*(\mathbb{G}^\nu) > 0$

Theorem Suppose that $\mathbb{G}$ is an infinite graph of bounded degree and $\iota_E^*(\mathbb{G}) > 0$. If ν has an exponential tail, then $\iota_E^*(\mathbb{G}^\nu) > 0$ a.s.

D. Chen & Y. Peres, Anchored expansion, percolation and speed, Ann. of Probability, (2004), Vol.32, No.4, 2978-2995

A Counter-Example.

If ν has a tail that decays slower than exponentially, then taking the binary tree as $\mathbb{G}$, we have $\iota_E^*(\mathbb{G}) > 0$ yet $\iota_E^*(\mathbb{G}^\nu) = 0$ a.s.

Corollary. For a supercritical Galton-Watson tree $\mathbb{T}$, given non-extinction we have $\iota_E^*(\mathbb{T}) > 0$ a.s.

III. The Critical value of percolation model

(a) Bernoulli bond percolation, Every edge is open with probability p,closed with probability $1-p$, all edges are independent of each other.

There is a critical value p_c, $0 < p_c < 1$.
When $p > p_c$ there is an open cluster;
When $p < p_c$ all open clusters are finite.

General speaking, it is not easy to identify p_c.
It can be calculated when G are trees.

(b) Suppose the probability that e is open is p_e, p_e's are different. If $\{p_e, e \in E(G)\}$ are i.i.d, essentially every edge is open with probability $p = Ep_e$.

(c) For a random stretched graph, every edge is open with probability p, then an edge of the original graph is open with probability p_e. $\{p_e, e \in E(G)\}$ are i.i.d.,

$$P(p_e = p^k) = \nu_k.$$

$$Ep_e = \sum_k \nu_k p^k = f(p),$$

where $f(s) = \sum_k \nu_k s^k$ is the moment generating function of ν.

If p_c is the critical value of the original graph G, then $f(p_c)$ is the critical value of random stretched graph G^ν.

(d) What about the site percolation?

IV. Contact Processes (I): Critical Value

Graph G is given. Every vertex x is either healthy or infected, denoted by 0 or 1 respectively. State changes from time to time.

$1 \longrightarrow 0$ at rate 1;

$0 \longrightarrow 1$ at rate $\lambda\times$ the number of infected neighbors.

λ is the only parameter of the model,
and there is a critical value λ_c.

The larger λ is, more easily the infection spreads, more sites are infected;
On the other hand, the smaller λ is, the smaller the range of infections, eventually all sites are healthy.

In the supercritical phase there are two different limiting behaviors. It is possible that a fixed site is infected only finite many times, though the infected area increases globally.

$\lambda_2 = \inf\{\lambda; P($ the origin is infected infinitely many times$) > 0\}$.

$$0 \leq \lambda_c \leq \lambda_2 \leq \infty.$$

Every strict inequality or equation is non-trivial at all.
For Z^d, $\lambda_c = \lambda_2$;
For T_d, $\lambda_c < \lambda_2$;

Given graph G and probability distribution ν, there is a family of random stretched graphs $G^{\nu}(\omega)$.

For each random stretched graph $G^{\nu}(\omega)$, there are also two critical values $\lambda_c(\omega) \leq \lambda_2(\omega)$.

For regular graphs such as $G = Z^d, T_d$, $\lambda_c(\omega)$ and $\lambda_2(\omega)$ are constants almost surely, denoted by λ_c, λ_2 respectively.

Question: When $\lambda_c < \lambda_2$?

Stacey (1996) Fixed stretched trees.

He LI (2005), Some G-W trees, ν is the geometric distribution.

Question: For T_d, for all ν with exponential decay,$\lambda_c < \lambda_2$?

Conjecture: $\lambda_c < \lambda_2 \Longleftrightarrow \iota_E^*(G) > 0$.

An estimate of λ_1.

$G = Z^d$, fixed stretched $q_k = 1$, $\lambda(k)$= the critical value of contact process on the fixed stretched graph.

Then $\lambda(1) \leq \lambda(2) \leq \lambda(4) \leq \cdots \leq \lambda(2^n) \leq \cdots$.

Question 1.
$\lambda(1) \leq \lambda(2) \leq \lambda(3) \leq \cdots \leq \lambda(n) \leq \lambda(n+1) \leq \cdots$?

Question 2. For random stretch ν, $\lambda(\nu)$ is a constant. If $E\nu = k$, $\lambda(\nu) > \lambda(k)$?

V. Contact Processes (II): Linear Growth

Suppose initially only one site o is infected,

A_t = all sites which have been infected by time t.

Theorem (H.X. ZHOU, 2008): Consider the contact process on fixed stretched Z^d, there is a convex set C, for $\epsilon > 0$,

$$\lim_t P((1-\epsilon)tC \subset A_t \subset (1+\epsilon)tC) = 1.$$

Should be valid for the contact process on random stretched Z^d.

This is true for the Richardson's model on random stretched Z^d.

For T_d, Let $B_r = \{x; dist(x,o) \leq r\}$. There are constants $a < A$ such that

$$\lim_t P(B_{at} \subset A_t \subset B_{At}) = 1.$$

References:

Benjamini, I., Lyons, R. & Schramm, O. (1999) Percolation Perturbations in Potential Theory and Random Walks, in *Random walks and Discrete Potential Theory.* Cambridge Univ. Press, 56-84.

D. Chen & Y. Peres, Anchored expansion, percolation and speed, , Ann. of Probability, (2004), Vol.32, No.4, 2978-2995.

He Li, MS Thesis, Peking University, 2005.

H.X. Zhou, MS Thesis, Peking University, 2008.

Thank You

E-Mail: dayue@pku.edu.cn