Barta's formula for the principal eigenvalues of Schrödinger operators

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Outline of this talk:

1. The background of the question and some known results.

2. Generalized variational formula for Dirichlet form.

3. Barta's Formula for Schrödinger operators.

Definitions and Notations

1. Assume (X, m) is a measure space, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a symmetric Dirichlet form on $L^2(X, m)$. We define

$$\lambda = \inf\{\mathscr{E}(f, f) : f \in \mathscr{D}(\mathscr{E}), m(f^2) = 1\},$$

and call λ the principal eigenvalue of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

2. L is a Markovian infinitesimal generator, c is a function bounded from below. The operator

$$L_c := L - c$$

is called Schrödinger operator.

Known Results

Assume (X, m) is a Polish space, $(q(x), q(x, dy))$ is a m-symmetric jump process, and its corresponding Dirichlet form is defined as

$$\mathscr{E}(f, f) = \frac{1}{2}\int m(dx)q(x, dy)\left[f(y) - f(x)\right]^2 + \int m(dx)r(x)f(x)^2,$$

$$\mathscr{D}(\mathscr{E}) = \left\{f \in L^2 : \ f|_{\{q>n\}} = 0 \ for\ some\ n\right\}.$$

Theorem A(Chen,2000). $f \in \mathscr{D}^+(\mathscr{E})$, then

$$\mathscr{E}(f, f) = \sup_g \langle f^2/g, -\Omega g\rangle,$$

where g varies over all the bounded positive functions.

● Known Results

Theorem B(Chen,2000).

$$\lambda \geq \sup_{0<\phi\in\mathscr{D}(\mathscr{E})} \operatorname{ess\,inf}(-\Omega\phi/\phi).$$

● Known Results

Assume $\{X_t\}$ is a Markov process with cadlag path. C^ϕ and C_b^ϕ are the sets of all finely continuous functions and bounded finely continuous functions on X respectively.

Shiozawa and Takeda defined $(\tilde{A}, \mathscr{D}(\tilde{A}))$, the weak generator of $\{X_t\}$ as following. For $u \in C_b^\phi(X)$, if there exists $g \in C_b^\phi(X)$ such that

$$u(X_t) - u(X_0) - \int_0^t g(X_s)ds$$

is a martingale, then we write

$$\tilde{A}u = g,$$

and let $\mathscr{D}(\tilde{A})$ be the set of all u with above property.

● Known Results

Theorem C.(Shiozawa & Takeda,2005) For $f \in \mathscr{D}^+(\mathscr{E})$,

$$\mathscr{E}(f,f) = \sup\left\{\int_X \frac{-\tilde{A}u}{u+\varepsilon} f^2 dm : \varepsilon > 0, u \in \mathscr{D}^+(\tilde{A})\right\},$$

where $\mathscr{D}^+(\tilde{A}) = \left\{u \in \mathscr{D}(\tilde{A}) : u \geq 0\right\}$.

Theorem D.(Shiozawa & Takeda,2005)

$$\lambda \geq \sup_{\phi \in \mathscr{F}} \operatorname{ess\,inf}(-\tilde{\mathrm{A}}\phi/\phi),$$

where $\mathscr{F} = \{u \in \mathscr{D}(\tilde{A}) : \sup|u| < \infty, u > 0, -\tilde{A}u > 0\}$.

● Main Results

Instead of the operator $(\tilde{A}, \mathscr{D}(\tilde{A}))$ we mentioned just now, we now define $(\hat{A}, \mathscr{D}_{loc}(\hat{A}))$, the local generator of the process $\{X_t\}$.

For $u \in C^{\phi}(X)$, if there exists $g \in C^{\phi}(X)$ such that

$$u(X_t) - u(X_0) - \int_0^t g(X_s)ds$$

is a local martingale, then we write

$$\hat{A}u = g,$$

and let $\mathscr{D}_{loc}(\hat{A})$ be the set of u with above property .

Main Results

Theorem 1. For $f \in \mathscr{D}^+(\mathscr{E})$,

$$\mathscr{E}(f,f) = \sup\left\{\int_X \frac{-\hat{A}u}{u+\varepsilon} f^2 dm : \varepsilon > 0, u \in \mathscr{D}^+_{loc}(\hat{A})\right\}.$$

Theorem 2. $\lambda \geq \sup_{\phi \in \mathscr{F}_{loc}} \operatorname{ess\,inf}(-\hat{A}\phi/\phi)$, where $\mathscr{F}_{loc} = \{u \in \mathscr{D}_{loc}(\hat{A}) : \sup|u| < \infty, u > 0, -\hat{A}u > 0\}$.

● Barta's Formula for Schrödinger Operators

Assume V is a differentiable function on $\mathbb{R}^n$, and

$$\mu(dx) = e^{V(x)} dx,$$

is the measure on $\mathbb{R}^n$. We study the operator

$$L_c = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j + \sum_{i=1}^{n} b_i \partial_i - c,$$

where $a_{ij} \in C^2, b_i = \sum_{j=1}^{n} (a_{ij} \partial_j V + \partial_j a_{ij})$.

Theorem 3.

$$\lambda(L_c) \geq \sup_{u \in C^2_{++}} \inf \frac{-L_c u}{u},$$

where $C^2_{++} = \{f \in C^2 : f > 0\}$.

Comparison of Some Known Estimations

We consider the diffusion process on $[0, +\infty)$ with Dirichlet boundary at 0. Assume a is a strictly positive measurable function on $[0, +\infty)$, and b is measurable on $[0, +\infty)$,$C(x) := \int_0^x \frac{b}{a}$. We study the second differential operator

$$Lf(x) = a(x)f''(x) + b(x)f'(x),$$

and the reference measure $\mu(dx) = \dfrac{e^{C(x)}}{a(x)} dx$.

Estimation A.(Muchenhoupt)

$$\lambda(L) \geq (4B)^{-1},$$

where $B = \sup\limits_{x>0} \int_0^x e^{-C(y)} dy \int_x^{+\infty} \frac{e^{C(y)}}{a(y)} dy.$.

● Comparison of Some Known Estimations

Estimation B.(Chen)

$$\lambda \geq \sup_{f \in \mathscr{F}'} \inf_{x>0} I\!I(f)^{-1}(x),$$

where

$$\mathscr{F}' = \left\{f \in C[0,+\infty) : f(0) = 0, f|_{(0,+\infty)} > 0\right\},$$

$$I\!I(f)(x) = \frac{1}{f(x)} \int_0^x dy e^{-C(y)} \int_y^{+\infty} \frac{f e^C}{a}, \quad f \in \mathscr{F}',$$

When a is continuous, the equality holds.

● Comparison of Some Known Estimations

Conclusion: For the principal eigenvalue of L in dimension 1, Barta's formula is equivalent to Chen's variational estimation, and both barta's formula and Chen's estimation are better than Muchenhoupt's estimation.

●References

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[3] Fukushima M., Oshima Y. and Takeda M., Dirichlet Forms and Symmetric Processes, Walter de Gruyter, New York, 1994.

THANK YOU VERY MUCH