Boundary value problems of elliptic operators with measurable coefficients

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The operator

Our work is concerned with the Dirichlet boundary value problems for the elliptic operators of the following form:

$$\begin{aligned} L &= \frac{1}{2}\nabla\cdot(a\nabla) + b\cdot\nabla - \nabla\cdot(\widehat{b}\cdot) + q, \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} \\ &\qquad - \text{``}div(\widehat{b}\cdot)'' + q(x) \end{aligned} \tag{1}$$

in a d-dimensional Euclidean domain D, where $a = (a_{ij}) : R^d \to R^d \times R^d$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda^{-1} I_d \leq a(\cdot) \leq \lambda I_d \tag{2}$$

for some constant $\lambda \geq 1$, $b, \hat{b} : R^d \to R^d$ and $q : R^d \to R$ are measurable functions which could be singular and such that

$$I_D |b|^2 \in K_d, \quad I_D |\hat{b}|^2 \in K_d \quad \text{and} \quad I_D q \in K_d, \tag{3}$$

where K_d denotes the space of Kato class measures. The operator L is defined by the following quadratic form

$$\begin{aligned} Q(u,v) &= (-Lu, v) &(4)\\ &= \frac{1}{2} \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \\ &\quad - \sum_{i=1}^{d} \int_{R^d} b_i(x) \frac{\partial u}{\partial x_i} v(x) dx \\ &\quad - \sum_{i=1}^{d} \int_{R^d} \hat{b}_i(x) \frac{\partial v}{\partial x_i} u(x) dx \\ &\quad - \int_{R^d} q(x) u(x) v(x) dx. &(5) \end{aligned}$$

Introduce the following condition

$$\sum_{i=1}^{d} \int_{R^d} \hat{b}_i(x) \frac{\partial \phi}{\partial x_i} dx + \int_{R^d} q(x)\phi(x) dx \leq 0, \quad (6)$$

for all nonnegative function ϕ in $C_0^\infty(D)$. This condition is equivalent to

$$div(\hat{b}) - q \geq 0$$

in the sense of distribution.

The following theorem is due to Trüdinger.

Theorem[Trüdinger]. Let $f \in W^{1,2}(D)$ be bounded. Assume (6). Then there exists a unique weak solution to the Dirichlet boundary value problem

$$Lu = 0, \quad u|_{\partial D} = f. \quad (7)$$

Recall that u is said to be a weak solution to (7) if $Q(u, \phi) = 0$ for any ϕ in $C_0^\infty(D)$ and $u - f \in W_0^{1,2}(D)$.

Our first aim is to provide a probabilistic representation for the weak solution. It is not obvious how to do it because there is no Markov process associated with the operator L due to the appearance of $\hat{b}$. The theory of Dirichlet forms and time reversal play an important role in our approach. Our second objective is to use the probabilistic representation to extend Trüdinger's result and to study the regularities of the solutions.

The process

The process we will use is the process generated by the symmetric part of the operator L. Consider the following regular Dirichlet form

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx \\ D(\mathcal{E}) &= \{u \in L^2(R^d, dx); \frac{\partial u}{\partial x_i} \in L^2(R^d, dx)\}\end{aligned}$$

It is well known that there exists a diffusion process, denoted by $\{\Omega, \mathcal{F}, X_t, \theta_t, \gamma_t, P_x, x \in R^d\}$, associated with $(\mathcal{E}, D(\mathcal{E}))$, where $\Omega = C([0,\infty) \to R^d)$, X_t is the canonical coordinate process, θ_t, γ_t are the shift and reverse operators defined by

$$X_s(\theta_t(\omega)) = X_{t+s}(\omega), \tag{8}$$

$$X_s(\gamma_t(\omega)) = X_{t-s}(\omega), \quad s \leq t. \tag{9}$$

X_t is not a semimartingale in general. But for any $u \in D(\mathcal{E})$, the following Fukushima decomposition holds:

$$u(X_t) - u(X_0) = M_t^u + N_t^u, \qquad P_x - a.s., \tag{10}$$

where M_t^u is a $\mathcal{F}_t = \sigma(X_s, s \leq t)$ square integrable continuous martingale additive functional and N_t^u is a continuous process of zero energy. M^u is often referred as the martingale part of the decomposition and N^u is the zero energy part. By a localization argument one can show that

$$X_t = x + M_t + N_t, \qquad P_x - a.s., \tag{11}$$

where $M_t = (M_t^1, ..., M_t^d)$ is a $\mathcal{F}_t = \sigma(X_s, s \leq t)$ square integrable continuous martingale additive functional with

$$< M^i, M^j >_t = \int_0^t a_{ij}(X_s) ds, \tag{12}$$

and N_t is a continuous process of local zero energy.

Theorem 1. Let u be the weak solution of (7). Then following representation holds.

$$\begin{aligned} u(x) &= E_x\Big[f(X_{\tau_D})exp\Big(\int_0^{\tau_D}(a^{-1}b)^*(X_s)dM_s \\ &\quad +(\int_0^{\tau_D}(a^{-1}\hat{b})^*(X_s)dM_s)\circ\gamma_{\tau_D} \\ &\quad -\frac{1}{2}\int_0^{\tau_D}(b-\hat{b})a^{-1}(b-\hat{b})^*(X_s)ds \\ &\quad +\int_0^{\tau_D}q(X_s)ds\Big)\Big], \end{aligned} \tag{13}$$

where $\tau_D = \inf\{t > 0; X_t \in D^c\}$.

Key points of the proof. Choose a special sequence of smooth vectors $\hat{b}_n$ and smooth functions q_n that approximates $\hat{b}$ and q respectively. Let u_n denote the unique weak solution of the following Dirichlet problem.

$$L^n u_n = 0, \quad u_n|_{\partial D} = f,$$

where L_n is defined as

$$L^n = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{d}b_i(x)\frac{\partial}{\partial x_i} - \sum_{i=1}^{d}\widehat{b}_{ni}(x)\frac{\partial}{\partial x_i} - div(\widehat{b}_n) + q_n(x), \quad (14)$$

The proof involves the following steps.

Step 1. Prove a probabilistic representation for u_n.

$$\begin{aligned} u_n(x) = E_x\Big[f(X_{\tau_D})exp\Big(&\int_0^{\tau_D}(a^{-1}b)^*(X_s)dM_s \\ &+(\int_0^{\tau_D}(a^{-1}\widehat{b}_n)^*(X_s)dM_s)\circ\gamma_{\tau_D} \\ &-\frac{1}{2}\int_0^{\tau_D}(b-\widehat{b})a^{-1}(b-\widehat{b}_n)^*(X_s)ds \\ &+\int_0^{\tau_D}q_n(X_s)ds\Big)\Big] \end{aligned} \quad (15)$$

This can be done using Girsanov theorem.

Step 2. Prove that u_n converges to u weakly in $W^{1,2}_{loc}(D)$.

Step 3. Prove that the probabilistic representation of u_n converges to the corresponding representation for u.

As a byproduct of our representation, we get the following extension of Trüdinger's result;

Theorem 2. For $f \in C(\partial D)$, there is a unique weak solution to the Dirichlet boundary value problem (7) which admits the representation:

$$\begin{aligned} u(x) \;=\; & E_x\Big[f(X_{\tau_D})exp\Big(\int_0^{\tau_D}(a^{-1}b)^*(X_s)dM_s \\ & +(\int_0^{\tau_D}(a^{-1}\hat{b})^*(X_s)dM_s)\circ\gamma_{\tau_D} \\ & -\frac{1}{2}\int_0^{\tau_D}(b-\hat{b})a^{-1}(b-\hat{b})^*(X_s)ds \\ & +\int_0^{\tau_D}q(X_s)ds\Big)\Big] \qquad (16) \end{aligned}$$

Sketch of the proof. Let u be defined as in (16). Take a sequence $f_n \in W^{1,2}(D)$ that converges uniformly to f on ∂D. Define u_n as in (16) with f replaced by f_n. Then, it is easy to see that $u_n \to u$ uniformly. On the other hand, we can show that u_n converges weakly to u in $W^{1,2}_{loc}(D)$, and u is a weak solution to the Dirichlet boundary value problem (7).

Regularity at the boundary

In Trüdinger's paper, no regularities were established for weak solutions on the boundary of the domain. Using the probabilistic representation, we are able to prove the following

Theorem 3. Assume $f \in C(\partial D)$, $|\hat{b}|^2|_D \in L^p(D)$ for some $p > \frac{d}{2}$, and $|b|^2|_D$, $q|_D$ belong to the Kato class. Let u be the unique weak solution of the Dirichlet problem (7). Then

$$\lim_{x \to y, x \in D} u(x) = f(y) \qquad (17)$$

for $y \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$.

The general case

In this part, we will drop the condition

$$div(\widehat{b}) - q \geq 0$$

and give a general result on the existence and uniqueness of the Dirichlet boundary value problem. We start with elliptic operators which do not have the adjoint drift part $\widehat{b}$.

Let $h = (h_1(x), ..., h_d(x)) : R^d \to R^d$ be a measurable function such that $h \in L^p(R^d \to R^d)$ for some $p > d$. Let $\mu = \mu_1 - \mu_2$ be a signed measure such that $\mu \in K_d - K_d$. Consider

$$L_1 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} h_i(x)\frac{\partial}{\partial x_i} + \mu. \tag{18}$$

L_1 is determined by the following quadratic form

$$\begin{aligned} Q^{(1)}(u,v) &= (-L_1 u, v) \qquad (19)\\ &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx \\ &\quad - \sum_{i=1}^{d}\int_{R^d} h_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &\quad - \int_{R^d} u(x)v(x)\mu(dx). \qquad (20) \end{aligned}$$

Let A_t be the continuous additive functional whose Revuz measure is μ. $E_x^{Q^1}$ will stand for the expectation with respect to the diffusion measure $Q_x^1, x \in R^d$, defined by

$$\left.\frac{dQ_x^1}{dP_x}\right|_{\mathcal{F}_t} = H_t^1, \tag{21}$$

where

$$H_t^1 =$$

$$exp\left(\int_0^t (a^{-1}h)^*(X_s)dM_s - \frac{1}{2}\int_0^t ha^{-1}h^*(X_s)ds\right)$$

Consider the Dirichlet boundary value problem:

$$L_1 u_1 = 0, \quad u|_{\partial D} = f(x). \tag{22}$$

Theorem 4. Assume $f \in C(\partial D)$ and

$$E_x\left[exp\left(\int_0^{\tau_D}(a^{-1}h)^*(X_s)dM_s \right.\right.$$
$$\left.\left.-\frac{1}{2}\int_0^{\tau_D} ha^{-1}h^*(X_s)ds + A_{\tau_D}\right)\right] < \infty$$

for some $x \in D$. Then there exists a unique, continuous weak solution to the Dirichlet boundary value problem (22) which is given by

$$\begin{aligned} u_1(x) &= E_x\Big[f(X_{\tau_D}) exp\Big(\int_0^{\tau_D} (a^{-1}h)^*(X_s)dM_s \\ &\qquad -\frac{1}{2}\int_0^{\tau_D} ha^{-1}h^*(X_s)ds + A_{\tau_D}\Big)\Big] \\ &= E_x^{Q^1}[f(X_{\tau_D})e^{A_{\tau_D}}]. \end{aligned} \tag{23}$$

Key points of the proof. We need three steps.

Step 1. We show that u_1 defined by (23) is a weak solution to the Dirichlet boundary value problem.

Step 2. We show that

$$\lim_{x \to y, x \in D} u_1(x) = f(y) \tag{24}$$

for $y \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$.

Step 3. We prove the uniqueness. To this end, using the theory of Dirichlet forms we show that any bounded, continuous weak solution of the Dirichlet boundary value problem (22) admits the probabilistic representation (23).

Now let us go back to our original problem

$$Lu = 0, \quad u|_{\partial D} = f, \tag{25}$$

where

$$\begin{aligned} L &= \frac{1}{2}\nabla \cdot (a\nabla) + b \cdot \nabla - \nabla \cdot (\widehat{b}\cdot) + q, \\ &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} \\ &\quad - \text{``}div(\widehat{b}\cdot)'' + q(x) \end{aligned} \tag{26}$$

Here is the result.

Theorem 5. Assume $f \in C(\partial D)$, $|\widehat{b}| \in L^p(D)$ for $p > d$, and that $|b|^2|_D$ and $q|_D$ belong to the

Kato class. Moreover suppose that

$$E_x\Big[exp\Big(\int_0^{\tau_D}(a^{-1}b)^*(X_s)dM_s$$
$$+(\int_0^{\tau_D}(a^{-1}\widehat{b})^*(X_s)dM_s)\circ\gamma_{\tau_D}$$
$$-\frac{1}{2}\int_0^{\tau_D}(b-\widehat{b})a^{-1}(b-\widehat{b})^*(X_s)ds$$
$$+\int_0^{\tau_D}q(X_s)ds\Big)\Big]<\infty. \tag{27}$$

for some $x \in D$. There exists a unique bounded, continuous weak solution to the Dirichlet problem (7) which has the representation:

$$\begin{aligned} u(x) \;=\; & E_x\Big[f(X_{\tau_D})exp\Big(\int_0^{\tau_D}(a^{-1}b)^*(X_s)dM_s \\ & +(\int_0^{\tau_D}(a^{-1}\widehat{b})^*(X_s)dM_s)\circ\gamma_{\tau_D} \\ & -\frac{1}{2}\int_0^{\tau_D}(b-\widehat{b})a^{-1}(b-\widehat{b})^*(X_s)ds \\ & +\int_0^{\tau_D}q(X_s)ds\Big)\Big]. \end{aligned} \tag{28}$$

The main idea of proof. The crucial idea is to transform the solution of (7) to a solution for an operator without adjoint drift by a kind of h-transform. More precisely, put

$$\widehat{M}_t = \int_0^{\tau_D} (a^{-1}\widehat{b})^*(X_s)dM_s.$$

We prove that there exits a bounded, continuous function $v \in D(\mathcal{E})$ such that

$$\widehat{M}_t \circ \gamma_t = -\widehat{M}_t + N_t^v,$$

where N^v is the zero energy part of the Fukushima decomposition for $v(X_t)-v(X_0)$ and moreover, v satisfies the following equation in the sense of distribution:

$$div(a\nabla v) = -2div(\widehat{b}). \qquad (29)$$

Thus,

$$(\int_0^{\tau_D} (a^{-1}\widehat{b})^*(X_s)dM_s) \circ \gamma_{\tau_D}$$

$$= -\int_0^{\tau_D} (a^{-1}\widehat{b})^*(X_s)dM_s + N_{\tau_D}^v$$

$$= -\int_0^{\tau_D} (a^{-1}\widehat{b})^*(X_s)dM_s + v(X_{\tau_D}) - v(X_0) - M_{\tau_D}^v$$

$$= -\int_0^{\tau_D} (a^{-1}\widehat{b})^*(X_s)dM_s + v(X_{\tau_D}) - v(X_0)$$

$$-\int_0^{\tau_D} \nabla v(X_s) dM_s. \tag{30}$$

Hence, $u = e^{-v(x)} u_2(x)$, where

$$\begin{aligned} & u_2(x) \\ = \quad & E_x\bigg[f(X_{\tau_D}) exp(v(X_{\tau_D})) \times \\ & exp\bigg(\int_0^{\tau_D} (a^{-1}(b - \widehat{b} - a\nabla v))^*(X_s) dM_s \\ & -\frac{1}{2}\int_0^{\tau_D} (b - \widehat{b} - a\nabla v) a^{-1} (b - \widehat{b} - a\nabla v)^*(X_s) ds \\ & -\int_0^{\tau_D} < b - \widehat{b}, \nabla v > (X_s) ds \\ & +\frac{1}{2}\int_0^{\tau_D} (\nabla v) a (\nabla v)^*(X_s) ds \\ & +\int_0^{\tau_D} q(X_s) ds \bigg)\bigg]. \end{aligned} \tag{31}$$

Introduce

$$
\begin{aligned}
L_2 \;=\; & \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) \\
& + \sum_{i=1}^{d}\left(b_i(x) - \widehat{b}_i(x) - (a\nabla v)_i(x)\right)\frac{\partial}{\partial x_i} \\
& - < b - \widehat{b}, \nabla v > (x) + \frac{1}{2}(\nabla v)a(\nabla v)^*(x) \\
& + q(x). \qquad (32)
\end{aligned}
$$

By Theorem 4, u_2 is a weak solution to the Dirichlet boundary value problem:

$$
L_2 u_2 = 0, \quad u_2|_{\partial D} = f(x)e^{v(x)}.
$$

Therefore, for any $\psi \in W_0^{1,2}(D)$,

$$
\begin{aligned}
& Q^*(u_2, \psi) = (-L_2 u_2, \psi) \\
= \ & \frac{1}{2} \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial u_2}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \\
& - \sum_{i=1}^{d} \int_{R^d} \left(b_i(x) - \hat{b}_i(x) - (a \nabla v)_i(x) \right) \frac{\partial u_2}{\partial x_i} \psi dx \\
& - \int_{R^d} q(x) u_2(x) \psi dx. \\
& + \int_{R^d} < b - \hat{b}, \nabla v > (x) u_2(x) \psi dx \\
& - \frac{1}{2} \int_{R^d} (\nabla v) a (\nabla v)^*(x) u_2 \psi dx \\
= \ & 0. \qquad (33)
\end{aligned}
$$

Using this we can show that u is the unique, bounded continuous slution to the Dirichlet boundary problem (7).