Reversibility of Chordal SLE

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SLE

SLE is a shorthand for Schramm-Loewner evolution or stochastic Loewner evolution. It is introduced by Oded Schramm in 1999, which combines Karl Löwner's equation in 1923 in Complex Analysis with some stochastic driving term. SLE becomes a fast growing area in the past several years, and is developed by many people: Oded Schramm, Wendelin Werner, Gregory F. Lawler, Stanislav Smirnov, Stephen Rohde, Scott Sheffield, Julien Dubédat, Robert Bauer, M. Bauer, D. Bernard, and others.

Chordal SLE

There are several versions of SLE. The chordal SLE is defined by solving the chordal Loewner equation:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z, \tag{1}$$

with $U(t) = \sqrt{\kappa} B(t)$, where $\kappa > 0$ and $B(t)$ is a standard linear Brownian motion. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For any fixed $z \in \mathbb{H}$, (1) is an ODE in t with initial value. Let $\tau(z) > 0$ be such that $[0, \tau(z))$ is the maximal interval on which the solution $t \mapsto g_t(z)$ exists.

For $t \geq 0$, let $K_t = \{z \in \mathbb{H} : \tau(z) \leq t\}$, i.e., the set of $z \in \mathbb{H}$ such that $g_t(z)$ blows up before or at time t. So g_t is defined on $H_t := \mathbb{H} \setminus K_t$. It turns out that g_t maps H_t conformally (univalent analytic) onto $\mathbb{H}$, fixes ∞, and satisfies $g_t(z) = z + \frac{2t}{z} + O(1/z^2)$ as $z \to \infty$. The set K_t is a random fractal. And g_t^{-1} extends continuously to $\mathbb{R}$. Let $\beta(t) = g_t^{-1}(U(t))$. Then β is a continuous curve in $\overline{\mathbb{H}}$ that satisfies $\beta(0) = 0$ and $\lim_{t\to\infty} \beta(t) = \infty$. Such β is called a standard chordal SLE$_\kappa$ trace.

Chordal SLE

The κ in the driving function $U(t) = \sqrt{\kappa}B(t)$ can not be eliminated by changing variables. In fact, the SLE_κ trace behaves differently for different values of κ. If $\kappa \in [0, 4]$, then β has no self intersections, and stays inside $\mathbb{H}$ except for the initial point $\beta(0) = 0$. In this case, we have $K_t = \beta((0, t])$, and $H_t = \mathbb{H} \setminus \beta((0, t])$. If $\kappa > 4$, then β intersects itself and $\mathbb{R}$ infinitely often. In that case H_t is the unbounded component of $\mathbb{H} \setminus \beta((0, t])$, and $K_t = \mathbb{H} \setminus H_t$. The Hausdorff dimension of the trace is $\min\{1 + \kappa/8, 2\}$. There is another phase change at $\kappa = 8$. For $\kappa \geq 8$, every point in $\mathbb{H}$ will be visited by the trace.

Conformal Invariance

If $D \subset \mathbb{C}$ is a simply connected domain with two boundary points a and b. There is W that maps $\mathbb{H}$ conformally onto D such that $W(0) = a$ and $W(\infty) = b$. Then $W \circ \beta$ is called an SLE_κ trace in D from a to b.

From the definition, it is clear that SLE satisfies the conformal invariance property. This property is interesting if combined with the next property.

Domain Markov Property

We explain the domain Markov property (DMP) for the standard chordal SLE_κ trace. Let $\beta(t)$ be a standard SLE_κ trace. Let H_t be the unbounded component of $\mathbb{H} \setminus \beta((0,t])$. Suppose T is a finite stopping time, then conditioned on $\beta(t)$, $0 \le t \le T$, $\beta(T+t)$, $t \ge 0$, is an SLE_κ trace in H_t from $\beta(t)$ to ∞. In another word, this means that $(g_T(\beta(T+t)) - U(T), t \ge 0)$ has the same distribution as $(\beta(t), t \ge 0)$. Note that $z \mapsto g_T(z) - U(T)$ maps H_t conformally onto $\mathbb{H}$, fixes ∞, and takes $\beta(T)$ to 0. In fact, the conformal invariance and domain Markov property determines SLE up to the parameter $\kappa \ge 0$.

Lattice Models

SLE is interesting because on the one hand SLE is a rigorous math model, and is computable; on the other hand, SLE is conjectured or proved to be the scaling limits of some two dimensional lattice models. These convergence combined with conformal invariance of SLE imply that some lattice models satisfy conformal invariance in the scaling limit. Physicist have these conjectures for many years.

Lattice Models

So far, people have proved the following results. The interface of the critical site percolation on triangular lattice converges to SLE_6. The loop-erased random walk converges to SLE_2. The boundary of uniform spanning tree with certain boundary conditions converges to SLE_8. The contour line of Gaussian free field converges to SLE_4. The interface of FK Ising model at critical temperature converges to SLE_3. And it is conjectured that self-avoiding walk converges to $\mathrm{SLE}_{8/3}$.

Reversibility

The convergence of these lattice models imply that SLE_κ trace is reversible for $\kappa = 6, 2, 8, 4, 3, 8/3$. This means that if β is an SLE_κ trace in D from a to b, then the reversal of β has the law as an SLE_κ trace in D from b to a up to a time-change. The reversibility of $\mathrm{SLE}_{8/3}$ is proved using restriction measure.

S. Rohde and O. Schramm conjectured that chordal SLE_κ trace is reversible for any $\kappa \in [0, 8]$. There are some signs that show that this is not true for $\kappa > 8$.

The Goal

In this talk I will explain the proof of the reversibility of chordal SLE_κ trace when $\kappa \in (0, 4]$. The case $\kappa \in (4, 8)$ is still open now.

The main idea is to construct some coupling measure of two SLE processes. We will find a coupling of SLE_κ from a to b with another SLE_κ from b to a such that every point in one trace is almost surely visited by the other trace.

Some Observations

Suppose $\kappa \in (0,4]$. Let's see what will happen if we assume that SLE_κ trace is reversible. Let γ_1 be an SLE_κ trace in D from a to b. Let γ_2 be the reversal of γ_1. Then γ_2 is an SLE_κ trace in D from b to a. Suppose $\overline{t}_2$ is a stopping time w.r.t. γ_2. From DMP of SLE, conditioned on $\gamma_2(t)$, $0 \le t \le \overline{t}_2$, the part of γ_2 after $\overline{t}_2$ is an SLE_κ trace in $D \setminus \gamma_2((0,\overline{t}_2])$ from $\gamma_2(\overline{t}_2)$ to x_1. Using the reversibility again, we see that the part of γ_1 before it hits $\gamma_2((0,\overline{t}_2])$ is an SLE_κ trace in $D \setminus \gamma_2((0,\overline{t}_2])$ from x_1 to $\gamma_2(\overline{t}_2)$. A symmetric result holds for any stopping time $\overline{t}_1$ w.r.t. γ_1.

Setup

Let $x_1 \neq x_2 \in \mathbb{R}$. For $j = 1, 2$, let β_j be an SLE_κ trace in $\mathbb{H}$ from x_j to x_{3-j}. Suppose β_1 is independent of β_2. Let μ_j denote the law of β_j. Then $\mu = \mu_1 \times \mu_2$ is the joint law of β_1 and β_2.

Let γ_1 have the same law as β_1, and γ_2 be the reversal of γ_1. Let $\nu = (\nu_1, \nu_2)$ denote the joint law of γ_1 and γ_2.

From conformal invariance, we suffice to prove that γ_2 has the same distribution as β_2.

Pair of Hulls

Let HP denote the set of pairs of hulls (H_1, H_2) in $\mathbb{H}$ such that H_j contains a neighborhood of x_j in $\mathbb{H}$, $j = 1, 2$, and $\overline{H_1} \cap \overline{H_2} = \emptyset$. Let $T_j(H_j)$ denote the first time that β_j or γ_j hits $\overline{\mathbb{H} \setminus H_j}$. Then $T_j(H_j)$ is an $(\mathcal{F}^j_t)$-stopping time, where $(\mathcal{F}^j_t)$ is the filtration generated by β_j or γ_j. And we have $0 < T_j(H_j) < T_j$.

Absolute Continuity

Fix $(H_1, H_2) \in \mathrm{HP}$. It is reasonable to believe (and in fact it is true) that the joint law of γ_1 and γ_2 up to the time $T_1(H_1)$ and $T_2(H_2)$ is absolutely continuous w.r.t. the joint law of β_1 and β_2 up to the time $T_1(H_1)$ and $T_2(H_2)$. Let RN denote the Radon-Nikodym derivative.

Two-Dimensional Martingale

For $t_1, t_2 \in [0, \infty)$, define

$$M_{H_1,H_2}(t_1, t_2) = \mathbf{E}\,[RN|\mathcal{F}^1_{t_1\wedge T_1(H_1)} \times \mathcal{F}^2_{t_2\wedge T_2(H_2)}].$$

Then for any fixed t_1, $M_{H_1,H_2}(t_1, \cdot)$ is an $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_t)_{t\geq 0}$-martingale, and for any fixed t_2, $M_{H_1,H_2}(\cdot, t_2)$ is an $(\mathcal{F}^1_t \times \mathcal{F}^2_{t_2})_{t\geq 0}$-martingale. We call such M_{H_1,H_2} a two-dimensional martingale.

On the other hand, if we know the martingale M_{H_1,H_2}, then we can use β_1 and β_2 to construct the joint law of γ_1 and γ_2 up to $T_1(H_1)$ and $T_2(H_2)$.

Two-Dimensional Local Martingale

Let $\mathcal{D}$ denote the set of $(t_1, t_2) \in [0, T_1) \times [0, T_2)$ such that $\overline{\beta_1((0, t_1])} \cap \overline{\beta_2((0, t_2])} = \emptyset$. For $t_2 \in [0, T_2)$, define $T_1(t_2)$ be the maximal such that $(t, t_2) \in \mathcal{D}$ for $0 \le t < T_1(t_2)$. Similarly, we define $T_2(t_1)$ for $t_1 \in [0, T_1)$.

Theorem

There is a random positive continuous function $M(t_1, t_2)$ defined on $\mathcal{D}$ such that for any $(H_1, H_2) \in \mathrm{HP}$, the restriction of M to $[0, T_1(H_1)] \times [0, T_2(H_2)]$ agrees with the M_{H_1,H_2} that we are looking for.

Properties

More explicitly, M satisfies the following properties.
1. $M(t_1, 0) = M(0, t_2) = 1$ for any $t_1 \in [0, T_1)$ and $t_2 \in [0, T_2)$. This conditions is to ensure that the marginal law $\nu_j = \mu_j$.
2. For any $(H_1, H_2) \in \mathrm{HP}$, $\ln(M)$ is uniformly bounded on $[0, T_1(H_1)] \times [0, T_2(H_2)]$.
3. For any $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2 < T_2$, $M(t, \bar{t}_2)$, $0 \le t < T_1(\bar{t}_2)$, is an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t \ge 0}$-local martingale. A symmetric result holds for any $(\mathcal{F}_t^1)$-stopping time $\bar{t}_1 < T_1$.

And most importantly, we have

4. Fix $(H_1, H_2) \in$ HP. Suppose the joint law of α_1 and α_2 is absolutely continuous w.r.t. the joint law of β_1 and β_2 up to time $T_1(H_1)$ and $T_2(H_2)$, and the Radon-Nikodym derivative is $M(T_1(H_1), T_2(H_2))$, then for any $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2 \in [0, T_1(H_1)]$, conditioned on $\mathcal{F}^2_{\bar{t}_2}$, α_1 up to time $T_1(H_1)$ is a part of an SLE$_\kappa$ trace in $\mathbb{H} \setminus \alpha_2((0, \bar{t}_2])$ from x_1 to $\gamma_2(\bar{t}_2)$. A symmetric result holds for any $(\mathcal{F}_t^1)$-stopping time $\bar{t}_1 \in [0, T_1(H_1)]$.

Local Coupling

The above theorem is proved by directly constructing M and checking its properties. Once the formula of M is written down, there is no difficulty in computation. We mainly use Ito's formula and Girsanov's Theorem.

Using β_1, β_2, and M, we may construct the joint law of γ_1 and γ_2 up to $T_1(H_1)$ and $T_2(H_2)$, for any $(H_1, H_2) \in \text{HP}$. Now the problem is how to construct the global joint law of γ_1 and γ_2.

Simple Observations

Since β_1 is independent of β_2, so β_1 and β_2 do not have "many" intersections. On the other hand, γ_1 overlaps with γ_2. Thus the global joint law of γ_1 and γ_2 is not absolutely continuous w.r.t. that of β_1 and β_2. This means that we can not extend M to a two-dimensional martingale defined on $[0, T_1] \times [0, T_2]$. However, we may extend part of M to a two-dimensional martingale defined on $[0, T_1] \times [0, T_2]$, and preserve the information of M as much as we need. This is the theorem below.

Constructing New Martingales

Theorem

Let $(H_1^k, H_2^k) \in \mathrm{HP}$, $1 \le k \le n$. There is a bounded positive two-dimensional martingale M_* defined on $[0,\infty] \times [0,\infty]$ such that $M_*(t,0) = M_*(0,t) = 1$ for any $t \ge 0$; and $M_* = M$ on $[0, T_1(H_1^k)] \times [0, T_2(H_2^k)]$ for any $1 \le k \le n$.

Proof

Proof. Here we mainly explain the construction of M_*. Write T_j^k for $T_j(H_j^k)$, $j = 1, 2$, $1 \le k \le n$.

There is $S \subset \{1, \dots, n\}$ such that

$$\cup_{m \in S}[0, T_1^m] \times [0, T_2^m] = \cup_{k=1}^n [0, T_1^m] \times [0, T_2^m].$$

We may choose S that minimizes $\sum_{m \in S} m$. Then such S is unique. Note that S is a random set.

Proof

There is a map σ from $\{1, \dots, |S|\}$ to S such that

$$T_1^{\sigma(1)} < T_1^{\sigma(2)} < \cdots < T_1^{\sigma(|S|)};$$

$$T_2^{\sigma(1)} > T_2^{\sigma(2)} > \cdots > T_2^{\sigma(|S|)}.$$

Define $T_1^{\sigma(0)} = T_2^{\sigma(|S|+1)} = 0$ and $T_2^{\sigma(0)} = T_1^{\sigma(|S|+1)} = \infty$. Then $T_1^{\sigma(k)}$ and $T_2^{\sigma(l)}$, $0 \le k, l \le |S| + 1$ divide $[0, \infty]^2$ into $(|S| + 1)^2$ small rectangles.

Proof

Fix $(t_1, t_2) \in [0, \infty]^2$. There are $1 \le k_1 \le |S| + 1$ and $0 \le k_2 \le |S|$ such that

$$T_1^{\sigma(k_1-1)} \le t_1 \le T_1^{\sigma(k_1)}, \quad T_2^{\sigma(k_2+1)} \le t_2 \le T_2^{\sigma(k_2)}.$$

If $k_1 \le k_2$, let

$$M_*(t_1, t_2) = M(t_1, t_2).$$

It $k_1 \ge k_2 + 1$, define $M_*(t_1, t_2) =$

$$\frac{M(T_1^{\sigma(k_2)}, t_2) M(T_1^{\sigma(k_2+1)}, T_2^{\sigma(k_2+1)}) \cdots M(t_1, T_2^{\sigma(k_1)})}{M(T_1^{\sigma(k_2)}, T_2^{\sigma(k_2+1)}) \cdots M(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_1)})}.$$

Proof

Note that if $k_1 \le k_2$, then $t_1 \le T_1^{\sigma(k_1)} \le T_1^{\sigma(k_2)}$ and $t_2 \le T_2^{\sigma(k_2)}$. So $(t_1, t_2) \in [0, T_1^{\sigma(k_2)}] \times [0, T_2^{\sigma(k_2)}]$. Then $M_*(t_1, t_2) = M(t_1, t_2)$ ensures that $M_* = M$ on $[0, T_1^k] \times [0, T_2^k]$ for each $1 \le k \le n$. That $M_*(t, 0) = M_*(0, t) = 1$ for $t \ge 0$ follows from the similar property of M.

Proof

The proof that M_* is a two-dimensional martingale is not trivial. Here we only give a rough idea. Note that if t_2 is fixed then $M_*(\cdot, t_2)$ is defined piecewise. For $T_1^{\sigma(k_1-1)} \le t_1 \le T_1^{\sigma(k_1)}$, we have

$$\frac{M_*(t_1, t_2)}{M_*(T_1^{\sigma(k_1-1)}, t_2)} = \frac{M(t_1, T_2^{\sigma(k_2+1)})}{M(T_1^{\sigma(k_1-1)}, T_2^{\sigma(k_2+1)})}.$$

So roughly speaking, $M_*(t_1, t_2)$, $t_1 \in [T_1^{\sigma(k_1-1)}, T_1^{\sigma(k_1)}]$, is a martingale. Since $[0, \infty]$ is the union of $[T_1^{\sigma(k_1-1)}, T_1^{\sigma(k_1)}]$, so $M_*(t_1, t_2)$, $0 \le t_1 \le \infty$, is a martingale. $\square$

Coupling Measures

Let HP_* denote the set of $(H_1, H_2) \in \mathrm{HP}$ such that for $j = 1, 2$, H_j is a polygon whose vertices have rational coordinates. Let (H_1^k, H_2^k), $k \in \mathbb{N}$, be an enumeration of HP_*. For $n \in \mathbb{N}$, let M_*^n be the M_* in the above theorem with (H_1^k, H_2^k), $1 \le k \le n$, be as in the above enumeration.

For $n \in \mathbb{N}$, define $\nu^n = (\nu_1^n, \nu_2^n)$ such that $d\nu^n / d\mu = M_*^n(\infty, \infty)$. From the Markov property of M_n^*, $\mathbf{E}_\mu[M_*^n(\infty, \infty)] = 1$. So ν^n is a probability measure for each n. The condition $M_*^n(t, 0) = M_*^n(0, t) = 1$ implies that $\nu_j^n = \mu_j$, $j = 1, 2$.

Coupling Measures

We may choose a suitable topology such that (ν^n) contains a subsequence (ν^{n_k}) that converges to some probability measure $\nu = (\nu_1, \nu_2)$. Then for $j = 1, 2$, $\nu_j^{n_k} \to \nu_j$, so $\nu_j = \mu_j$. Thus ν is a coupling of β_1 and β_2.

Fix $m \in \mathbb{N}$. If $n_k \geq m$ then $M_*^{n_k} = M$ on $[0, T_1^m] \times [0, T_2^m]$. So

$$\frac{d\nu^{n_k}}{d\mu}\Big|_{\mathcal{F}^1_{T_1^m} \times \mathcal{F}^2_{T_2^m}} = M(T_1^m, T_2^m).$$

Here $T_j^m = T_j(H_j^m)$. Since this holds true for any $n_k \geq m$, so it still holds if ν^{n_k} is replaced by ν.

Coupling Measures

Suppose the joint law of α_1 and α_2 is ν. Then for any $m \in \mathbb{N}$ and any $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2 \le T_2^m$, conditioned on $\mathcal{F}_{\bar{t}_2}^2$, $\alpha_1(t)$, $0 \le t \le T_1^m$, has the law of a part of an SLE_κ trace in $\mathbb{H} \setminus \alpha_2((0, \bar{t}_2])$ from x_1 to $\alpha_2(\bar{t}_2)$. This follows from the property of M we have discussed.

Coupling Measures

Now fix an $(\mathcal{F}_t^2)$-stopping time $\bar{t}_2 < T_2$. For $n \in \mathbb{N}$ define

$$R_n = \sup\{T_1^m : 1 \le m \le n, T_2^m \ge \bar{t}_2\}.$$

Here we set $\sup \emptyset = 0$. Then R_n is an $(\mathcal{F}_t^1 \times \mathcal{F}_{\bar{t}_2}^2)_{t \ge 0}$-stopping time. For $1 \le m \le n$, let $\bar{t}_2^m = t_2 \wedge T_2^m$. Then $\bar{t}_2^m \le T_2^m$ is an $(\mathcal{F}_t^2)$-stopping time. Thus $\alpha_1(t)$, $0 \le t \le T_1^m$, has the law of a part of an SLE$_\kappa$ trace in $\mathbb{H} \setminus \alpha_2((0, \bar{t}_2^m])$ from x_1 to $\alpha_2(\bar{t}_2^m)$. Let

$$\mathcal{E}_{n,m} = \{\bar{t}_2 \le T_2^m\} \cap \{R_n = T_1^m\}.$$

Then we have $\{R_n > 0\} = \cup_{m=1}^n \mathcal{E}_{n,m}$.

Coupling Measures

Since on each $\mathcal{E}_{n,m}$, we have $T_1^m = R_n$ and $\bar{t}_2^m = \bar{t}_2$, so $\alpha_1(t)$, $0 \le t \le R_n$, has the law of a part of an SLE_κ trace in $\mathbb{H} \setminus \alpha_2((0, \bar{t}_2])$ from x_1 to $\alpha_2(\bar{t}_2)$.

We have

$$T_1(\bar{t}_2) = \sup\{T_1^m : m \in \mathbb{N}, \bar{t}_2 \le T_2^m\} = \vee_{n=1}^{\infty} R_n.$$

Thus $\alpha_1(t)$, $0 \le t < T_1(\bar{t}_2)$, has the law of a part of an SLE_κ trace in $\mathbb{H} \setminus \alpha_2((0, \bar{t}_2])$ from x_1 to $\alpha_2(\bar{t}_2)$.

From the definition of $T_1(\bar{t}_2)$, $\alpha_1([0, T_1(\bar{t}_2)])$ intersects $\alpha_2([0, \bar{t}_2])$. From the property of SLE$_\kappa$ trace for $\kappa \in (0, 4]$ we have $\alpha_1(T_1(\bar{t}_2)) = \alpha_2(\bar{t}_2)$ a.s.. Thus $\alpha_2(\bar{t}_2)$ is a.s. visited by α_1.

We may choose a sequence of $(\mathcal{F}_t^2)$-stopping times $(\bar{t}_2^n)$ on $[0, T_2)$ such that $\{\bar{t}_2^n : n \in \mathbb{N}\}$ is dense on $[0, T_2)$. Then a.s. α_1 visits each $\alpha_2(\bar{t}_2^n)$. So a.s. α_1 visits every point on α_2. Thus $\alpha_2 \subset \alpha_1$. Similarly, $\alpha_1 \subset \alpha_2$. So $\alpha_1 = \alpha_2$. Then we are done.