A central limit type theorem for particle filter

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Outline

1. Introduction
2. Filtering equations
3. Branching particle system
4. Convergence of V_t^n
5. Convergence of V^n
6. A central limit type theorem

1. Introduction

Filtering problem:

signal:

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \\ &\quad + \int_0^t c(X_s)dW_s \end{aligned} \tag{1}$$

with **observation**

$$Y_t = \int_0^t h(X_s)ds + W_t \tag{2}$$

where on $(\Omega, \mathcal{F}, P_0)$, W_t and B_t are indep. B.M. of dimesions d and m.

Information available at time $\boldsymbol{t}$:

$$\boldsymbol{\mathcal{G}_t = \sigma(Y_s : \ s \leq t).}$$

Optimal filter $\boldsymbol{\pi_t}$ is $\boldsymbol{\mathcal{P}}(\mathbf{R}^{\mathbf{d}})$-valued process:

$$\boldsymbol{\langle \pi_t, f \rangle} = \mathbf{E}^{\mathbf{P}_0}(\mathbf{f}(\mathbf{X}_\mathbf{t})|\boldsymbol{\mathcal{G}}_\mathbf{t})$$

.

2. Filtering equations

Let P be given by

$$\frac{dP_0}{dP} = \exp\left(\int_0^T h(X_s)dY_s - \frac{1}{2}\int_0^T |h(X_s)|^2 ds \right).$$

Girsanov theorem: Under P, Y is a B.m., indep. of B.

Signal:

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t \tilde{b}(X_s)ds \\ &\quad + \int_0^t c(X_s)dY_s \end{aligned}$$

where

$$\tilde{b}(x) = b(x) - ch(x).$$

Kallianpur-Striebel formula:

$$\langle \pi_t, f \rangle = \mathrm{E}^{P_0}\left(\mathrm{f}(\mathrm{X}_t)|\mathcal{G}_t\right) = \frac{\langle \mathrm{V}_t, \mathrm{f} \rangle}{\langle \mathrm{V}_t, 1 \rangle}, \qquad \forall\ \mathrm{f} \in \mathrm{C_b}(\mathrm{R}^d)$$

where

$$\langle V_t, f \rangle = \mathrm{E}\left(\mathrm{M}_t \mathrm{f}(\mathrm{X}_t)|\mathcal{G}_t\right)$$

and

$$dM_t = M_t h(X_t) dY_t.$$

Assumption (BC): $\boldsymbol{\sigma}, \ \boldsymbol{b}, \ \boldsymbol{c}, \ \boldsymbol{h}$ *are Lipschitz and uniformly bounded.*

Assumption (I): As $\boldsymbol{n \to \infty}$,

$$\boldsymbol{V_0^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i^n} \to \pi_0.}$$

Thm (Kurtz-X): Suppose (BC) and (I) hold. Then

$$\langle V_t, f\rangle = \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} M_t^i f(X_t^i), \tag{3}$$

where $\{(M^i, X^i):\ i = 1, 2, \cdots\}$ unique strong solution to

$$\begin{aligned} X_t^i &= X_0^i + \int_0^t \sigma(X_s^i)dB_s^i + \int_0^t \tilde{b}(X_s^i)ds \\ &\quad + \int_0^t c(X_s^i)dY_s \end{aligned}$$

and

$$dM_t^i = M_t^i h(X_t^i)dY_t, \qquad M_0^i = 1. \tag{4}$$

Thm: (Zakai equation)

$$d\langle V_t, f\rangle = \langle V_t, Lf\rangle\, dt + \left\langle V_t, \nabla^* f c + f h^*\right\rangle dY_t. \qquad (5)$$

Proof: By Itô's formula, we have

$$df(X_t^i) = \tilde{L}f(X_t^i) + \nabla^* f\sigma(X_t^i)dB_t^i + \nabla^* fc(X_t^i)dY_t.$$

Hence

$$\begin{aligned} d(M_t^i f(X_t^i)) &= M_t^i Lf(X_t^i)dt + M_t^i \nabla^* f\sigma(X_t^i)dB_t^i \\ &\quad + M_t^i \nabla^* fc(X_t^i)dY_t + M_t^i fh^* dY_t. \end{aligned}$$

Thm: (Kushner-FKK equation)

$$d\langle \pi_t, f\rangle = \langle \pi_t, Lf\rangle\, dt$$
$$+ \left(\langle \pi_t, \nabla^* f c + f h^*\rangle - \langle \pi_t, f\rangle \langle \pi_t, h^*\rangle\right) d\nu_t$$

where

$$\nu_t = Y_t - \int_0^t \langle \pi_s, h\rangle\, ds$$

is the innovation process.

Proof: Applying Itô's formula, we get

$$
\begin{aligned}
d\langle \pi_t, f\rangle &= \frac{1}{\langle V_t, 1\rangle}\left(\langle V_t, Lf\rangle dt + \left\langle V_t, \nabla^* fc + fh^*\right\rangle dY_t\right) \\
&\quad - \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle^2}\left\langle V_t, h^*\right\rangle dY_t \\
&\quad - \frac{1}{\langle V_t, 1\rangle^2}\left\langle V_t, \nabla^* fc + fh^*\right\rangle \langle V_t, h\rangle dt \\
&\quad + \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle^3}\left\langle V_t, h^*\right\rangle \langle V_t, h\rangle dt.
\end{aligned}
$$

3. Branching particle system

At $t = 0$, n particles of weight $\frac{1}{n}$ each at $x_i^n,\ i = 1, 2, \cdots, n$.

Let $\delta = \delta_n = n^{-2\alpha}$, $0 < \alpha < 1$.

At $t = j\delta$, m_j^n particles alive. For $t \in (j\delta, (j+1)\delta)$, particles move by

$$X_t^i = X_0^i + \int_0^t \sigma(X_s^i)dB_s^i + \int_0^t \tilde{b}(X_s^i)ds + \int_0^t c(X_s^i)dY_s.$$

At $t = (j+1)\delta$, ith particle ($i = 1, 2, \cdots, m_j^n$) branches (independent of others) into ξ_{j+1}^i offsprings

$$\mathbf{E}\left(\xi_{j+1}^i \middle| \mathcal{F}_{(j+1)\delta-}\right) = \tilde{M}_{j+1}^n(X^i)$$

and

$$Var\left(\xi_{j+1}^i \middle| \mathcal{F}_{(j+1)\delta-}\right) = \gamma_{j+1}^n(X^i)$$

where

$$\tilde{M}_{j+1}^n(X^i) = \frac{M_{j+1}^n(X^i)}{\frac{1}{m_j^n}\sum_{\ell=1}^{m_j^n} M_{j+1}^n(X^\ell)} \tag{6}$$

and

$$M_{j+1}^{n}(X^{i}) \tag{7}$$
$$= \exp\left(\int_{j\delta}^{(j+1)\delta} h^{*}(X_t^{i})dY_t - \frac{1}{2}\int_{j\delta}^{(j+1)\delta} |h(X_t^{i})|^2 dt\right).$$

To minimize γ_{j+1}^{n}, we take

$$\xi_{j+1}^{i} = \begin{cases} [\tilde{M}_{j+1}^{n}(X^{i})] & \text{w.p. } 1 - \{\tilde{M}_{j+1}^{n}(X^{i})\} \\ [\tilde{M}_{j+1}^{n}(X^{i})] + 1 & \text{w.p. } \{\tilde{M}_{j+1}^{n}(X^{i})\} \end{cases}$$

where $\{x\} = x - [x]$ is the fraction of x. In this case

$$\gamma_{j+1}^{n}(X^{i}) = \{\tilde{M}_{j+1}^{n}(X^{i})\}(1 - \{\tilde{M}_{j+1}^{n}(X^{i})\}).$$

Define

$$\tilde{\pi}_t^n = \frac{1}{n} \sum_{i=1}^{m_j^n} \delta_{X_t^i}, \qquad j\delta \le t < (j+1)\delta.$$

$\tilde{\pi}_t^n$ is particle filter (Crisan-Del Moral-Lyons, etc).

Define

$$\pi_t^n = \sum_{i=1}^{m_j^n} \tilde{M}_j^n(X^i, t)\delta_{X_t^i}, \qquad j\delta \le t < (j+1)\delta$$

where

$$M_j^n(X^i, t) = \exp\left(\int_{j\delta}^t h^*(X_s^i)dY_s - \frac{1}{2}\int_{j\delta}^t |h(X_s^i)|^2 ds\right). \tag{8}$$

π_t^n is hybrid filter (Crisan-X).

Define $\boldsymbol{V_t^n = \pi_t^n \eta_t^n}$, where

$$\boldsymbol{\eta_t^n = \Pi_{j=0}^k \frac{1}{m_j^n} \sum_{\ell=1}^{m_j^n} M_{j+1}^n(X^\ell),} \qquad \text{if } \boldsymbol{k\delta \le t < (k+1)\delta.}$$

We will prove that $\boldsymbol{V_t^n}$ converges to the unnormalized filter $\boldsymbol{V_t}$.

4. Convergence of V_t^n

Dual of V is sol. to Backward SPDE:

$$\begin{cases} d\psi_s = -L\psi_s ds - (\nabla^*\psi_s c + h\psi_s)\,\hat{d}Y_s, & 0 \le s \le t \\ \psi_t = \phi. \end{cases} \tag{9}$$

Lemma: Assume *(BD):*

$$a_{ij},\ \ b_i\ \ h \in C_b^{1+[d/2]}(\mathrm{R^d})$$

and

$$\partial_\alpha \phi \in H_0 \equiv L^2(\mathrm{R^d}), \quad \forall |\alpha| \le 1 + [\mathrm{d}/2].$$

Then

$$\mathbf{E}\left(\|\psi_s\|_0^2 + \sum_{|\alpha|\le 1+[d/2]} \|\partial_\alpha \psi_s\|_0^2 \right) \le K_1 \tag{10}$$

where $\|f\|_0^2 = \int_{\mathrm{R^d}} |f(x)|^2 dx$. As a consequence, $\psi_s \in C(\mathrm{R^d})$ a.s.

Let $k\delta \le t < (k+1)\delta$. Then

$$
\begin{aligned}
& \left\langle \mathbf{V}_t^n, \phi \right\rangle - \left\langle \mathbf{V}_0^n, \psi_0 \right\rangle \\
= \; & \left\langle \mathbf{V}_t^n, \psi_t \right\rangle - \left\langle \mathbf{V}_{k\delta}^n, \psi_{k\delta} \right\rangle \\
& + \sum_{j=1}^{k} \left(\left\langle \mathbf{V}_{j\delta}^n, \psi_{j\delta} \right\rangle - \mathbf{E}\left(\left\langle \mathbf{V}_{j\delta}^n, \psi_{j\delta} \right\rangle | \mathcal{F}_{j\delta-} \vee \mathcal{G}_{j\delta,t} \right) \right) \\
& + \sum_{j=1}^{k} \left(\mathbf{E}\left(\left\langle \mathbf{V}_{j\delta}^n, \psi_{j\delta} \right\rangle | \mathcal{F}_{j\delta-} \vee \mathcal{G}_{j\delta,t} \right) \right. \\
& \quad \left. - \left\langle \mathbf{V}_{(j-1)\delta}^n, \psi_{(j-1)\delta} \right\rangle \right) \\
\equiv \; & I_1^n + I_2^n + I_3^n,
\end{aligned}
\tag{11}
$$

where

$$I_1^n = \eta_{k\delta}^n \frac{1}{n} \sum_{i=1}^{m_k^n} \Big(M_k^n(X^i, t)\psi_t(X_t^i) - \psi_{k\delta}(X_{k\delta}^i) \Big),$$

$$I_2^n = \sum_{j=1}^{k} \eta_{j\delta}^n \frac{1}{n} \sum_{i=1}^{m_{j-1}^n} \psi_{j\delta}(X_{j\delta}^i)(\xi_j^i - \tilde{M}_j^n(X^i))$$

and

$$I_3^n = \sum_{j=1}^{k} \eta_{(j-1)\delta}^n \frac{1}{n} \sum_{i=1}^{m_{j-1}^n} \Big(\psi_{j\delta}(X_{j\delta}^i) M_j^n(X^i) - \psi_{(j-1)\delta}(X_{(j-1)\delta}^i) \Big).$$

Lemma: [Crisan-Gaines-Lyons]

$$\psi_{(j+1)\delta}(X^i((j+1)\delta))M^n_{j+1}(X^i) - \psi_{j\delta}(X^i(j\delta))$$
$$= \int_{j\delta}^{(j+1)\delta} M^n_j(X^i, s)\nabla^*\psi_s\sigma(X^i_s)dB^i_s, \quad (12)$$

where

$$M^n_j(X^i, s) = \exp\left(\int_{j\delta}^{s} h(X^i_t)dY_t - \frac{1}{2}\int_{j\delta}^{s} |h(X^i_t)|^2_H dt\right).$$

As a consequence,

$$\langle V_t, \phi\rangle = \langle \pi_0, \psi_0\rangle .$$

Thm:

$$\mathbf{E}|\left\langle \boldsymbol{V_t^n}, \boldsymbol{\phi}\right\rangle - \left\langle \boldsymbol{V_t}, \boldsymbol{\phi}\right\rangle|^2 \le \boldsymbol{K_1 n^{-(1-\alpha)}}.$$

Similarly,

$$\mathbf{E}| < \boldsymbol{\tilde{V}_t^n}, \boldsymbol{\phi} > - < \boldsymbol{V_t}, \boldsymbol{\phi} > |^2 \le \boldsymbol{K_2}\left(\boldsymbol{n^{-(1-\alpha)}} \vee \boldsymbol{n^{-2\alpha}}\right).$$

Remark: For particle filter $\boldsymbol{\tilde{V}_t^n}$ the optimal $\boldsymbol{\alpha}$ is $\boldsymbol{\frac{1}{3}}$. Best rate is $\boldsymbol{n^{-1/3}}$.

5. Convergence of V^n

Key equation:

$$\begin{aligned} \langle V_t^n, f\rangle &= \langle V_0^n, f\rangle + \int_0^t \langle V_s^n, Lf\rangle\, ds \\ &\quad + \int_0^t \langle V_s^n, \nabla^* f c + hf\rangle\, dY_s + N_t^{n,f} + \hat{N}_t^{n,f}, \end{aligned} \tag{13}$$

where

$$N_t^{n,f} = \sum_{j=0}^{[t/\delta]} \frac{1}{n} \sum_{i=1}^{m_j^n} \int_{j\delta}^{((j+1)\delta)\wedge t} \nabla^* f\sigma(X_s^i)\, dB_s^i \eta_{j\delta}^n$$

$$\hat{N}_t^{n,f} = \sum_{j=1}^{[t/\delta]} \eta_{j\delta}^n \frac{1}{n} \sum_{i=1}^{m_{j-1}^n} (\xi_j^i - \tilde{M}_j^n(X^i)) f(X_{j\delta}^i).$$

are uncorrelated w/

$$\left\langle N^{n,f}\right\rangle_t = \sum_{j=0}^{[t/\delta]}\frac{1}{n^2}\sum_{i=1}^{m_j^n}\int_{j\delta}^{((j+1)\delta)\wedge t}|\nabla^* f\sigma(X_s^i)|^2 ds(\eta_{j\delta}^n)^2$$

$$\left\langle \hat{N}^{n,f}\right\rangle_t = \sum_{j=1}^{[t/\delta]}\frac{1}{n^2}\sum_{i=1}^{m_{j-1}^n}\gamma_j^n(X^i)f^2(X_{j\delta}^i)(\eta_{j\delta}^n)^2. \tag{14}$$

Define the usual distance

$$d(\nu_1, \nu_2) = \sum_{k=1}^{\infty} 2^{-k} (| \langle \nu_1 - \nu_2, f_k \rangle | \wedge 1)$$

where $f_k \in C_b^2(\mathrm{R}^{\mathrm{d}})$ with $f_k, \ \partial_j f_k, \ \partial_{ij}^2 f_k, i, j = 1, 2, \cdots, d$, bounded by 1.

Thm:

$$\mathbf{E} \sup_{\mathbf{t} \leq \mathbf{T}} \mathbf{d}(\mathbf{V}_\mathbf{t}^\mathbf{n}, \mathbf{V}_\mathbf{t})^2 \leq \mathbf{K}_1 \mathbf{n}^{-(1-\alpha)}.$$

Similarly

$$\mathbf{E} \sup_{\mathbf{t} \leq \mathbf{T}} \mathbf{d}(\tilde{\mathbf{V}}_\mathbf{t}^\mathbf{n}, \mathbf{V}_\mathbf{t})^2 \leq \mathbf{K}_2 \left(\mathbf{n}^{-2\alpha} \vee \mathbf{n}^{-(1-\alpha)} \right).$$

6. A central limit type theorem

Let

$$\boldsymbol{U_t^n = n^{\frac{1-\alpha}{2}}(V_t^n - V_t), \quad t \geq 0.}$$

Then

$$\begin{aligned}\langle U_t^n, f\rangle &= \langle U_0^n, f\rangle + \int_0^t \langle U_s^n, Lf\rangle\, ds \\ &\quad + \int_0^t \langle U_s^n, \nabla^* f c + hf\rangle\, dY_s \\ &\quad + n^{\frac{1-\alpha}{2}} N_t^{n,f} + n^{\frac{1-\alpha}{2}} \hat{N}_t^{n,f}, \end{aligned} \tag{15}$$

Let $\psi(x) \approx e^{-|x|}$ smooth. Let

$$\Phi = \{\phi : \phi\psi \in \mathcal{S}\},$$

where $\mathcal{S} = Schwarz$. Define

$$\|\phi\|_\kappa^2 = \sum_{0 \le |k| \le \kappa} \int_{\mathrm{R}} (1 + |x|^2)^{2\kappa} \left| \frac{\partial^k}{\partial x^k} (\phi(x)\psi(x)) \right|^2 dx.$$

Let $\Phi_\kappa = \overline{\Phi}$ w.r.t. $\|\cdot\|_\kappa$. Let $\Phi_{-\kappa}$ be the dual.

Thm: $\exists \kappa$ s.t. $\{U^n\}$ is tight in $D_{\Phi_{-\kappa}}[0, \infty)$.

Finally we characterize the limit.

Thm: $U^n \Longrightarrow U$ which is the unique sol. to

$$\begin{aligned}
& \langle U_t, f\rangle - \langle U_0, f\rangle \\
= \; & \int_0^t \langle U_s, Lf\rangle\, ds + \int_0^t \langle U_s, \nabla^* f c + hf\rangle\, dY_s \\
& + \sqrt[4]{\frac{2}{\pi}} \int_0^t \int_{\mathrm{R^d}} \sqrt{|h(x) - \pi_s h| V_s(x) \langle V_s, 1\rangle} f(x) B(ds dx).
\end{aligned}$$