Intersection Properties of Lévy Processes

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1. Lévy processes

A Lévy process $Z = \{Z(t),\ t \geq 0\}$ in $\mathbb{R}^d$ is a process with stationary and independent increments, $X(0) = 0$ a.s. and is continuous in probability.

For $t \geq s \geq 0$, the characteristic function of $Z(t) - Z(s)$ is given by

$$\mathbb{E}\left[e^{i\langle \xi, Z(t)-Z(s)\rangle}\right] = e^{-(t-s)\Psi(\xi)}.$$

By the Lévy-Khintchine formula,

$$\Psi(\xi) = i\langle \mathsf{a}, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma\xi^{'}\rangle + \int_{\mathbb{R}^d}\left[1 - e^{i\langle x,\xi\rangle} + \frac{i\langle x, \xi\rangle}{1+\|x\|^2}\right] \mathsf{L}(dx)$$

for all $\xi \in \mathbb{R}^d$.

In the above, a $\in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $d \times d$ matrix, and L is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{\|x\|^2}{1+\|x\|^2}\,\mathsf{L}(dx) < \infty.$$

Ψ is called the Lévy exponent of Z, and L is the corresponding Lévy measure.

Example: If $\Psi(\xi) = c\,\|\xi\|^\alpha$ ($\alpha \in (0,2]$), then Z is called an isotropic stable Lévy process with index α. Its Lévy measure L is of the form

$$\mathsf{L}(dx) = \frac{dr}{r^{1+\alpha}}\,\nu(dy), \quad \forall\, x = ry, \;\; (r,y) \in \mathbb{R}_+ \times \mathbb{S}_d,$$

where $\nu(dy)$ is the uniform measure on $\mathbb{S}_d = \{y \in \mathbb{R}^d : \;\|y\| = 1\}$.

2. Intersections of Lévy processes

Let $X_1, \ldots, X_k$ be k independent Lévy processes with values in $\mathbb{R}^d$. Consider the random sets

$$M_k = X_1\big((0,\infty)\big) \cap \cdots \cap X_k\big((0,\infty)\big),$$

$$L_k = \Big\{(t_1, \cdots, t_k) \in (0,\infty)^k : \; X_1(t_1) = \cdots = X_k(t_k)\Big\}.$$

We can ask the following questions:

(i). When does $M_k \neq \emptyset$?

(ii). Given a Borel set $F \subseteq \mathbb{R}^d$, when does $\mathbb{P}\{M_k \cap F \neq \varnothing\} > 0$?

(iii). If $M_k \neq \emptyset$, what are the Hausdorff dimensions of M_k and L_k?

3. Three approaches for studying Question (i)

a. Classical results

- Question (i) was first studied for Brownian motion in $\mathbb{R}^d$ by Dvoretzky, Erdös and Kakutani (1950, 1954, 1958) and Dvoretzky, Erdös, Kakutani and Taylor (1957).
- Later their results were extended by Takeuchi (1964) and Taylor (1966) to isotropic stable Lévy processes, and by Taylor (1967) to strictly stable processes of type A.
- Hendricks (1974, 1979) considered Question (i) for Lévy processes with stable components.

All the above papers use potential theory of Lévy processes.

b. Intersection local times

Let $Z = \{Z(t),\ t \in \mathbb{R}_+^k\}$ be defined by

$$Z(t) = \big(X_2(t_2) - X_1(t_1), \ldots, X_k(t_k) - X_{k-1}(t_{k-1})\big).$$

Note that $M_k \neq \varnothing$ is equivalent to $Z^{-1}(0) \neq \varnothing$.

Hence Question (i) can be studied by the existence of intersection local times.

- Geman, Horowitz and Rosen (1984), Dynkin (1985), Rosen (1983, 1987) for Brownian motion.
- Le Gall, Rosen and Shieh (1989) for Lévy processes.
- See also Khoshnevisan, Shieh and Xiao (2007).

Using this approach, one can also prove results on the Hausdorff and packing dimensions of L_k and M_k.

c. Potential theory for multiparameter processes

Consider the product Lévy process $Y = \{Y(t), t \in \mathbb{R}_+^k\}$ defined by

$$Y(t) = \big(X_1(t_1), \ldots, X_k(t_k)\big).$$

Denote $D = \{(x, \ldots, x) \in \mathbb{R}^{kd}\}$. Note that

$$M_k \neq \varnothing \quad \text{if and only if} \quad Y(\mathbb{R}_+^k) \cap D \neq \varnothing.$$

By studying hitting probabilities of Y, Evans (1987a, b) showed that, if $X_1, \ldots, X_k$ are i.i.d., $v^1(0) > 0$ and

$$\int_{\|x\| \le 1} \left[v^1(x)\right]^k dx < \infty, \tag{1}$$

then $\mathbb{P}\{M_k \neq \varnothing\} > 0$. Here v^1 is the 1-potential density of X_1.

Fitzsimmons and Salisbury (1989) further proved that, if $X_1, \ldots, X_k$ are i.i.d. and $v^1(0) > 0$, then (1) is also necessary for $M_k \neq \varnothing$.

This verifies the Hendricks–Taylor conjecture.

Recently Khoshnevisan and Xiao (2007) proved potential theoretical results for the additive Lévy processes $X = \{X(t), t \in \mathbb{R}_+^k\}$ defined by

$$X(t) = X_1(t_1) + \cdots + X_k(t_k).$$

Their results lead to answers to Questions (ii) and (iii).

4. Potential theory for additive Lévy processes

First we seek to answer the following question:

Given a Borel set $F \subseteq \mathbb{R}^d$, when can $\mathbb{P}\{X(\mathbb{R}_+^k) \cap F \neq \emptyset\} > 0$?

For all finite measures μ on $\mathbb{R}^d$, the energy of μ is

$$\mathcal{E}_\Psi(\mu) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi)}\right) d\xi,$$

where $\widehat{\mu}$ denotes the Fourier transform of μ.

For all Borel sets $F \subseteq \mathbb{R}^d$, the Ψ-capacity of F is defined as

$$\mathrm{cap}_{\boldsymbol{\Psi}}(F) := \left[\inf_{\mu \in \mathcal{P}_c(F)} I_{\boldsymbol{\Psi}}(\mu)\right]^{-1}.$$

The following theorem was proved in Khoshnevisan, Xiao and Zhong (2003) under an extra condition, and in Khoshnevisan and Xiao (2007) in general.

Theorem 4.1. Let X be a (k, d)-additive Lévy process with Lévy exponent $\Psi = (\Psi_1, \ldots, \Psi_k)$. Then, for any nonrandom compact set $F \subseteq \mathbb{R}^d$, $\mathbb{E}\{\lambda_d(X(\mathbb{R}_+^k) \oplus F)\} > 0$ if and only if F carries a finite measure μ such that $\mathcal{E}_\Psi(\mu) < \infty$.

When $k = 1$, Kesten (1969) proved the above theorem for $F = \{0\}$ and Hawkes (1984) for general F.

We say that X has a 1-potential density $v(x)$ if

$$\mathbb{E}\left[\int_{\mathbb{R}_+^k} f(X(t))\, e^{-(t_1+\cdots+t_k)}\, dt\right] = \int_{\mathbb{R}^d} f(x)v(x)\, dx$$

for all Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$.

Theorem 4.2. If X has an a.e.-positive 1-potential density, then the following statement are equivalent:

(i) $\mathbb{E}\Big[\lambda_d\big(X(\mathbb{R}_+^k) \oplus F\big)\Big] > 0$;

(ii) $\mathbb{P}\left\{X\big((0\,,\infty)^k\big) \cap F \neq \varnothing\right\} > 0$;

(iii) $\mathrm{cap}_{\mathbf{\Psi}}(F) > 0$.

Corollary 4.3 Suppose $X_1, \ldots, X_k$ are independent isotropic stable processes in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$. Then for any Borel set $F \subset \mathbb{R}^d$, the following are equivalent:

(i) $\mathrm{cap}_{d-k\alpha}(F) > 0$;

(ii) $\mathbb{P}\big\{\lambda_d\{F \oplus X(\mathbb{R}^k_+)\} > 0\big\} > 0$;

(iii) F is not polar for X in the sense that
$\mathbb{P}\big\{F \cap X(\mathbb{R}^k_+ \backslash \{0\}) \neq \emptyset\big\} > 0$.

Combined with Frostman's theorem, this result provides a useful way for Hausdorff dimension computation: For any $F \subseteq \mathbb{R}^d$,

$$\dim F = d - \inf\big\{k\alpha > 0 : \; F \; \text{ is not polar for } X^{\alpha,k}\big\},$$

where $X^{\alpha,k}$ is the additive stable process in Corollary 4.3.

Product Lévy processes

Now we consider the "product Lévy process" $\otimes_{j=1}^{k} X_j$ defined by

$$\left(\otimes_{j=1}^{k} X_j\right)(t) := \begin{pmatrix} X_1(t_1) \\ \vdots \\ X_k(t_k) \end{pmatrix} \qquad \text{for all} \quad t \in \mathbb{R}_+^k.$$

Observation: The process $\otimes_{j=1}^{k} X_j$ is a degenerate additive Lévy process.

Indeed, for all $t \in \mathbb{R}_+^k$ we can write

$$\left(\otimes_{j=1}^{k} X_j\right)(t) = \mathbf{A}^1 X_1(t_1) + \cdots + \mathbf{A}^k X_k(t_k),$$

where each $X_j(t_j)$ is viewed as a column vector of dimension d and each $\mathbf{A}^j$ is the $kd \times d$ matrix

$$\mathbf{A}^j = \begin{pmatrix} \mathbf{0}_{(j-1)d\times d} \\ \mathbf{I}_{d\times d} \\ \mathbf{0}_{(k-j)d\times d} \end{pmatrix}.$$

Theorems 4.1 and 4.2 yield

Theorem 4.4 Let $X_1, \ldots, X_k$ be independent Lévy processes on $\mathbb{R}^d$, and assume that each X_j has a one-potential density u_j such that $u_j(0) > 0$. Then, for all Borel sets $D \subseteq \mathbb{R}^{kd}$,

$$\mathbb{P}\left\{ \left(\otimes_{j=1}^k X_j \right) \left((0\,,\infty)^k \right) \cap D \neq \varnothing \right\} > 0$$

if and only if there exists a Borel probability measure μ on D such that

$$\int_{\mathbb{R}^{kd}} \left| \hat{\mu}(\xi^1, \ldots, \xi^k) \right|^2 \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi^j)} \right) d\xi < \infty.$$

5. Hitting probabilities of M_k

Theorem 5.1 Let $X_1, \ldots, X_k$ be independent Lévy processes on $\mathbb{R}^d$, and assume that each X_j has a one-potential density u_j such that $u_j(0) > 0$. Then for any Borel set $F \subseteq \mathbb{R}^d$,

$$\mathbb{P}\Big\{X_1(t_1) = \cdots = X_k(t_k) \in F \text{ for some } t_1, \ldots, t_k > 0\Big\} > 0$$

if and only if there exists a Borel probability measure μ on F such that

$$\int_{\mathbb{R}^{kd}} \left|\hat{\mu}\left(\xi^1 + \cdots + \xi^k\right)\right|^2 \prod_{j=1}^{k} \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi^j)}\right) d\xi^1 \cdots d\xi^k < \infty.$$

Proof. Let $D = \{(x, \ldots, x) : x \in F\}$. The conclusion follows from Theorem 4.4.

6. The Hausdorff dimension of M_k

Theorem 6.1 Let $X_1, \ldots, X_k$ be independent Lévy processes on $\mathbb{R}^d$, and assume that each X_j has a one-potential density u_j such that $u_j(0) > 0$. Then, almost surely on $\{M_k \neq \varnothing\}$,

$$\dim_{\mathrm{H}} M_k = \sup \Bigg\{ s \in (0\,, d) : \int_{(\mathbb{R}^d)^k} \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi^j)} \right) \frac{d\xi}{1 + \|\xi^1 + \cdots + \xi^k\|^{d-s}} < \infty \Bigg\},$$

where $\sup \varnothing := 0$.

Proof. The proof is based on Theorem 4.2 and Corollary 4.3.

Theorem 6.2 Suppose, in addition, that the u_j's are continuous on $\mathbb{R}^d$, and finite on $\mathbb{R}^d \setminus \{0\}$. Then, almost surely on $\{M_k \neq \varnothing\}$,

$$\dim_{\mathrm{H}} M_k = \sup\left\{ s \in (0\,, d) : \int_{\mathbb{R}^d} \prod_{j=1}^{k} \left(\frac{u_j(z) + u_j(-z)}{2} \right) \frac{dz}{\|z\|^s} < \infty \right\}.$$

7. An Example

Suppose $X_1, \ldots, X_k$ are independent isotropic stable processes in $\mathbb{R}^d$ with indices $\alpha_1, \ldots, \alpha_k \in (0, 2]$, respectively.

It follows from Theorems 5.1 and 6.1 that for all Borel sets $F \subseteq \mathbb{R}^d$,

$$\mathbb{P}\{M_k \cap F \neq \varnothing\} > 0 \quad \Leftrightarrow \quad \mathrm{cap}_{kd - \sum_{j=1}^k \alpha_j}(F) > 0,$$

and almost surely on $\{M_k \neq \varnothing\}$,

$$\dim_{\mathrm{H}} M_k = \left[\sum_{j=1}^k \alpha_j - (k-1)d\right]_+ .$$

8. On Bertoin's conjecture

Let $S_1 = \{S_1(t_1), t_1 \geq 0\}$ and $S_2 = \{S_2(t_2), t_2 \geq 0\}$ be two subordinators starting at 0. Then the closed ranges $\overline{S_1(\mathbb{R}_+)}$ and $\overline{S_2(\mathbb{R}_+)}$ are regenerative sets.

They were studied by many authors such as Krylov and Yushkevich (1965), Kingman (1972), Hoffmann-Jørgensen (1969), Maisonneuve (1971, 1974), Hawkes (1977), Fristedt (1996) and Bertoin (1999a, 1999b).

Now we consider the intersection problem of S_1 and S_2. Assume that S_1 has a continuous 0-potential density u_1^0 which is positive a.e. on $\mathbb{R}_+$ and let U_2^0 be the 0-potential measure of S_2.

Bertoin (1999a) proved that

$$\mathbb{P}\left\{S_1(t_1) = S_2(t_2) \quad \text{for some } t_1, t_2 > 0\right\} = 0$$

$$\Leftrightarrow \quad \sup_{z \in \mathbb{R}} \int_0^\infty u_1^0(y+z)\, U_2^0(dy) = \infty.$$

By using Theorem 5.1, we can prove that

$$\mathbb{P}\left\{S_1(t_1) = S_2(t_2) \quad \text{for some } t_1, t_2 > 0\right\} = 0$$

$$\Leftrightarrow \quad \int_0^\infty u_1^0(y)\, U_2^0(dy) = \infty.$$

This verifies a conjecture of Bertoin (1999b).

THANK YOU