Spectral gap for quasi-birth-death processes with application to Jackson networks

Liang-Hui XIA

Beijing Normal University

xlhui2004@163.com

July 16, 2007

(joint with Yong-Hua MAO)

Content

- Introduction of quasi-birth-death processes (QBD)
- Decomposition method
- Spectral gap for QBD
- Ergodicity of Jackson networks
- Spectral gap for Jackson networks

QBD

- Quasi-birth and death process (QBD) is the process on the state space $X = \{(i,k) : i \in \mathbb{Z}_+, k \in E_i\}$, here $\mathbb{Z}_+$ is the level set, and E_i is the phase set.

QBD

- Quasi-birth and death process (QBD) is the process on the state space $X = \{(i,k) : i \in \mathbb{Z}_+, k \in E_i\}$, here $\mathbb{Z}_+$ is the level set, and E_i is the phase set.
- The Q-matrix of QBD is as follows

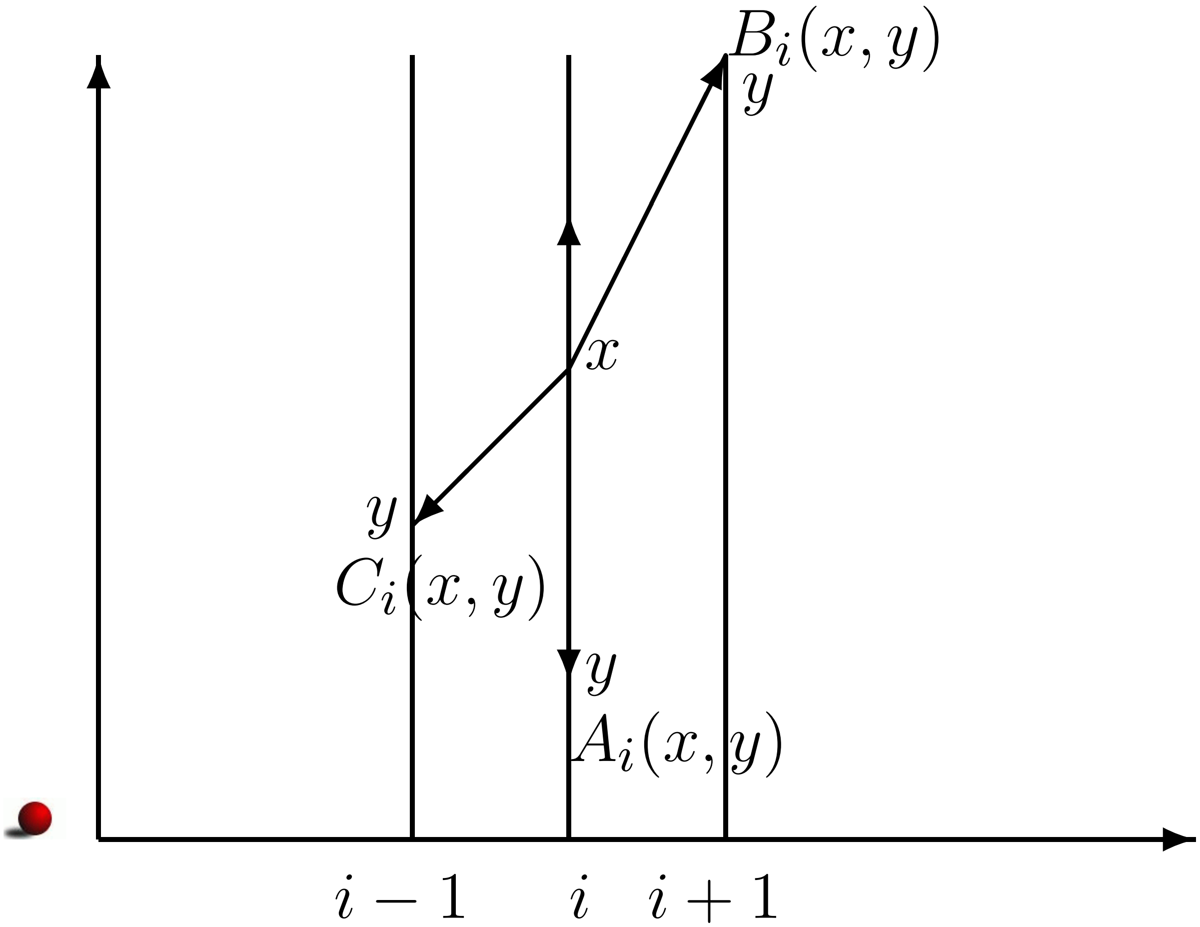
$$Q = (q(\alpha,\beta)) = \begin{pmatrix} A_0 & B_0 & & & \\ C_1 & A_1 & B_1 & & \\ & C_2 & A_2 & B_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$A_i(x,y) = q(i,x;i,y), \quad x \in E_i, y \in E_i;$$

$$B_i(x,y) = q(i,x;i+1,y), \quad x \in E_i, y \in E_{i+1};$$

$$C_i(x,y) = q(i,x;i-1,y), \quad x \in E_i, y \in E_{i-1}.$$

QBD



QBD

- For finite phase set, and A_k, B_k, C_k are same for $k \geq k_0$, Neuts [1981] has developed the matrix-geometric meth -od to study the ergodicity of QBD, and get its invariant probability measure. And in this case, ergodicity is equivalent to exponential ergodicity(Hou and Li [2005, 2007]), but no one study the convergence rate.

QBD

- For finite phase set, and A_k, B_k, C_k are same for $k \geq k_0$, Neuts [1981] has developed the matrix-geometric meth -od to study the ergodicity of QBD, and get its invariant probability measure. And in this case, ergodicity is equivalent to exponential ergodicity(Hou and Li [2005, 2007]), but no one study the convergence rate.
- Recently, for infinite phase set, and A_k, B_k, C_k are same for $k \geq k_0$, under some conditions, one can also get the same results by matrix-geometric method. (Kroese, et al.[2004, Ann.Appl.Prob.])

QBD

- For finite phase set, and A_k, B_k, C_k are same for $k \geq k_0$, Neuts [1981] has developed the matrix-geometric meth -od to study the ergodicity of QBD, and get its invariant probability measure. And in this case, ergodicity is equivalent to exponential ergodicity(Hou and Li [2005, 2007]), but no one study the convergence rate.
- Recently, for infinite phase set, and A_k, B_k, C_k are same for $k \geq k_0$, under some conditions, one can also get the same results by matrix-geometric method. (Kroese, et al.[2004, Ann.Appl.Prob.])
- For general QBD like Q-matrix defined before, we can't analyze it by matrix-geometric method.

QBD

- Suppose Q is regular, irreducible, and reversible with respect to probability measure π, for $i \geq 0$,

$$\pi(i,x)A_i(x,y) = \pi(i,y)A_i(y,x), \quad x,y \in E_i;$$

$$\pi(i,x)B_i(x,y) = \pi(i+1,y)C_{i+1}(y,x), \quad x \in E_i, y \in E_{i+1}.$$

QBD

- Suppose Q is regular, irreducible, and reversible with respect to probability measure π, for $i \geq 0$,

$$\pi(i,x)A_i(x,y) = \pi(i,y)A_i(y,x), \quad x, y \in E_i;$$

$$\pi(i,x)B_i(x,y) = \pi(i+1,y)C_{i+1}(y,x), \quad x \in E_i, y \in E_{i+1}.$$

- Aim: To estimate the spectral gap for QBD,

$$D(f) = \frac{1}{2}\sum_{\alpha,\beta \in X} \pi(\alpha)q(\alpha,\beta)(f(\alpha) - f(\beta))^2,$$

$$\mathrm{gap}(Q) = \inf\{D(f) : f \in \mathcal{D}(D), \pi(f) = 0, \pi(f^2) = 1\}.$$

Decomposition method

- Decomposition method has been used in Jerrum et al. [2004, Ann.Appl.Prob.] and Madras et al. [2002, Ann. Appl.Prob.] to estimate the spectral gap and Log-Sobo -lev constant for finite Markov chains. In our paper, we also use the decomposition method to estimate the spectral gap for QBD.

Decomposition method

- Decomposition method has been used in Jerrum et al. [2004, Ann.Appl.Prob.] and Madras et al. [2002, Ann. Appl.Prob.] to estimate the spectral gap and Log-Sobo -lev constant for finite Markov chains. In our paper, we also use the decomposition method to estimate the spectral gap for QBD.
- Decompose π along the level set $\pi = (\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, ...)$, where $\pi^{(i)}$ is the restriction of π on the level i, and $\pi_i(x) = \xi_i^{-1}\pi^{(i)}$ is a probability on E_i, here ξ_i is the normalizati -on constant.

Decomposition method

- Decompose the Dirichlet form of QBD along the level:

$$
\begin{aligned}
D(f) \; = \; & \sum_{k=0}^{\infty} \xi_k \frac{1}{2} \sum_{x,y \in E_k} \pi_k(x) A_k(x,y) (f_k(x) - f_k(y))^2 \\
& + \sum_{k=0}^{\infty} \xi_k \sum_{x \in E_k, y \in E_{k+1}} \pi_k(x) B_k(x,y) (f_k(x) - f_{k+1}(y))^2
\end{aligned}
$$

Decomposition method

- Decompose the Dirichlet form of QBD along the level:

$$D(f) = \sum_{k=0}^{\infty} \xi_k \frac{1}{2} \sum_{x,y \in E_k} \pi_k(x) A_k(x,y)(f_k(x) - f_k(y))^2 + \sum_{k=0}^{\infty} \xi_k \sum_{x \in E_k, y \in E_{k+1}} \pi_k(x) B_k(x,y)(f_k(x) - f_{k+1}(y))^2$$

- The variance can be also decomposed

$$\mathrm{Var}(f) = \sum_{k=0}^{\infty} \xi_k \mathrm{Var}_k(f_k) + \sum_{k=0}^{\infty} \xi_k (\mathbb{E}_k f_k - \mathbb{E} f)^2$$

Decomposition method

- Comparing $D(f), \mathrm{Var}(f)$, we can construct a sequence Dirichlet forms restricted on each level and a birth -death process .

Decomposition method

- Comparing $D(f), \mathrm{Var}(f)$, we can construct a sequence Dirichlet forms restricted on each level and a birth -death process .
- Define Dirichlet form on E_i

$$D_i(f_i) = \frac{1}{2} \sum_{x,y \in E_i} \pi_i(x) A_i(x,y)(f_i(x) - f_i(y))^2$$

$$\delta_i = \inf\{D_i(f_i) : f_i \in \mathcal{D}(D_i), \pi_i(f_i) = 0, \pi_i(f_i^2) = 1\}$$

Decomposition method

- Comparing $D(f), \mathrm{Var}(f)$, we can construct a sequence Dirichlet forms restricted on each level and a birth -death process .
- Define Dirichlet form on E_i

$$D_i(f_i) = \frac{1}{2} \sum_{x,y \in E_i} \pi_i(x) A_i(x,y)(f_i(x) - f_i(y))^2$$

$$\delta_i = \inf\{D_i(f_i) : f_i \in \mathcal{D}(D_i), \pi_i(f_i) = 0, \pi_i(f_i^2) = 1\}$$

- Define a birth-death process (BD) $\overline{Q}$,

$$b_i = \sum_{x \in E_i, y \in E_{i+1}} \pi_i(x) B_i(x,y) \qquad (i \geq 0)$$

$$a_i = \sum_{x \in E_i, y \in E_{i-1}} \pi_i(x) C_i(x,y) \qquad (i \geq 1)$$

Decomposition method

- Notations: $d_k = \inf_{x\in E_k} \sum_{y\in E_{k+1}} B_k(x,y)$,
$e_k = \inf_{x\in E_k} \sum_{y\in E_{k-1}} C_k(x,y)$,
$c_k = \sup_{x\in E_k} [\sum_{y\in E_{k+1}} B_k(x,y) + \sum_{z\in E_{k-1}} C_k(x,z)]$,

$$\alpha_k = c_k - \frac{d_k^2}{b_k} - \frac{e_k^2}{a_k}$$

where we use the convention $C_0(x,y) = 0, e_0 = 0$.

Decomposition method

- Notations: $d_k = \inf_{x\in E_k} \sum_{y\in E_{k+1}} B_k(x,y)$,
$$e_k = \inf_{x\in E_k} \sum_{y\in E_{k-1}} C_k(x,y),$$
$$c_k = \sup_{x\in E_k} [\sum_{y\in E_{k+1}} B_k(x,y) + \sum_{z\in E_{k-1}} C_k(x,z)],$$
$$\alpha_k = c_k - \frac{d_k^2}{b_k} - \frac{e_k^2}{a_k}$$
where we use the convention $C_0(x,y) = 0, e_0 = 0$.
- α_k plays an important role in our estimation, and in some cases, $\alpha_k = 0$ for all $k \geq 0$, and the spectral gap estimation can be sharp, which we will see in Corollary 2.

Spectral gap for QBD

- For the spectral gap of BD $\overline{Q}$, Chen's book [2004] presented perfect solutions, which can be expressed by a variational formula. Denote $\overline{\lambda} = \text{gap}(\overline{Q})$.

Spectral gap for QBD

- For the spectral gap of BD $\overline{Q}$, Chen's book [2004] presented perfect solutions, which can be expressed by a variational formula. Denote $\overline{\lambda} = \mathrm{gap}(\overline{Q})$.
- Theorem 1: If $\delta_k > 0$ for all $k \geq 0$, then

$$\frac{\overline{\lambda}}{\sup_{k \geq 0} q_k} \leq \mathrm{gap}(Q) \leq \overline{\lambda},$$

where $\{q_k\}$ is a sequence of numbers greater than or equals to 1.

Spectral gap for QBD

- For the spectral gap of BD $\overline{Q}$, Chen's book [2004] presented perfect solutions, which can be expressed by a variational formula. Denote $\overline{\lambda} = \text{gap}(\overline{Q})$.
- Theorem 1: If $\delta_k > 0$ for all $k \geq 0$, then

$$\frac{\overline{\lambda}}{\sup\limits_{k\geq 0} q_k} \leq \text{gap}(Q) \leq \overline{\lambda},$$

  where $\{q_k\}$ is a sequence of numbers greater than or equals to 1.
- In case that $\alpha_k = 0$, if $\delta_k < \overline{\lambda}$, $q_k = \frac{\overline{\lambda}}{\delta_k}$; if $\delta_k \geq \overline{\lambda}$, $q_k = 1$.

Spectral gap for QBD

- Corollary 2:
  Case 1. If $B_k = b_k\mathbf{I}, C_k = a_k\mathbf{I}$, where $\mathbf{I}$ is a unit matrix, which is finite or infinite.
  Case 2. If $B_k = b_k\mathbf{1}\mu^T$, $C_k = a_k\mathbf{1}\nu^T$, where **1**, μ, ν are column vector, and every component of **1** is 1
  Then: (1) If $\delta_k \geq \overline{\lambda}$ for all k, we have

$$\mathrm{gap}(Q) = \overline{\lambda};$$

(2) If $\delta_k < \overline{\lambda}$ for all k, we have

$$\inf_{k\geq 0} \delta_k \leq \mathrm{gap}(Q) \leq \overline{\lambda}.$$

Open Jackson networks

- N-node open Jackson network is a continuous-time random walk on $\mathbb{Z}_+^N$ with transition intensities $q_{\alpha\beta}$, from the state $\alpha = (\alpha^1, ..., \alpha^N)$ to the state $\beta = (\beta^1, ..., \beta^N)$, where

$$q_{\alpha\beta} = \begin{cases} \lambda_i, & \text{if } \beta - \alpha = e_i \\ \mu_i p_{i0}, & \text{if } \beta - \alpha = -e_i \\ \mu_i p_{ij}, & \text{if } \beta - \alpha = -e_i + e_j \end{cases} \qquad 1 \le i, j \le N$$

Here e_i denote the vector $(0, ..., 0, 1, 0, ..., 0)$, having its i-th coordinate equal to 1.

Open Jackson networks

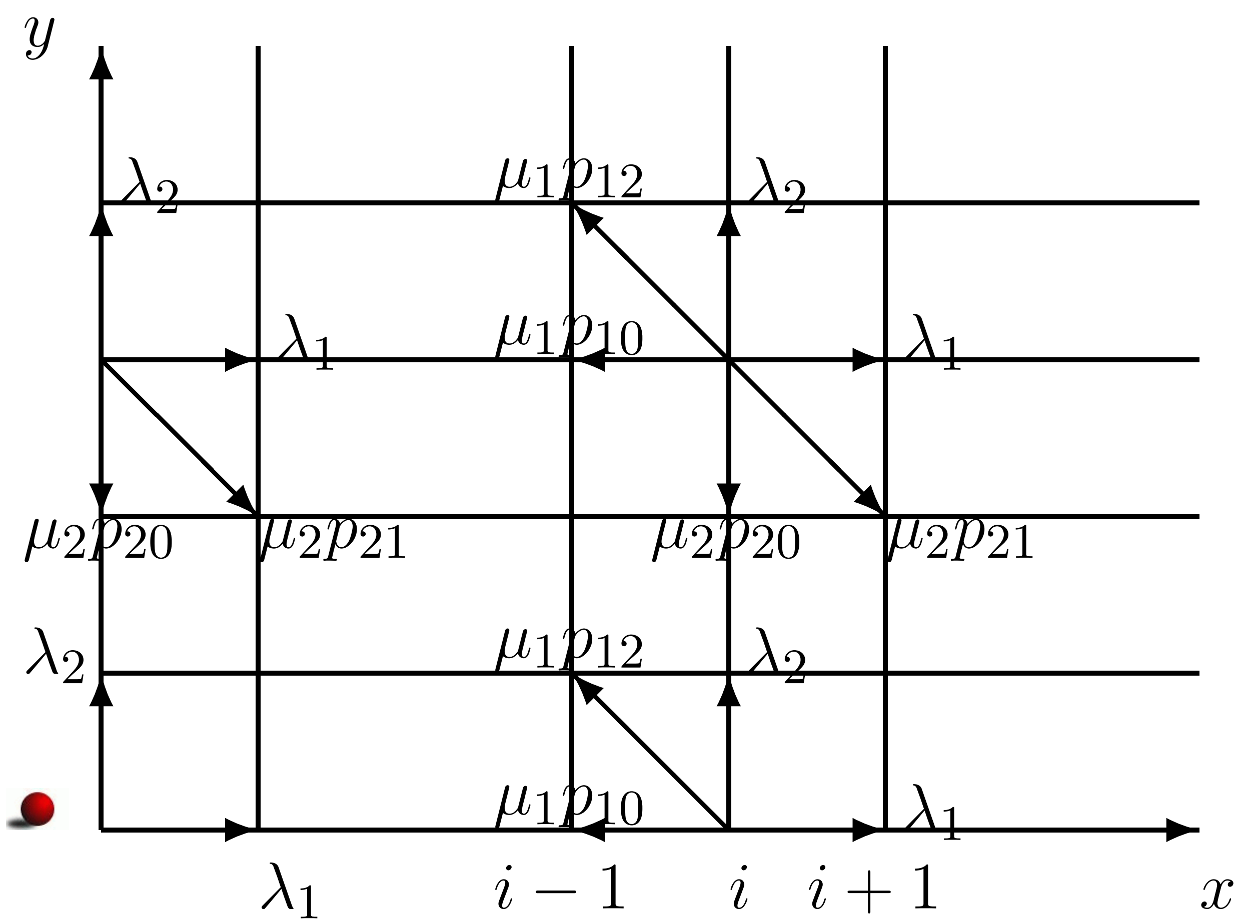
- N-node open Jackson network is a continuous-time random walk on $\mathbb{Z}_+^N$ with transition intensities $q_{\alpha\beta}$, from the state $\alpha = (\alpha^1, ..., \alpha^N)$ to the state $\beta = (\beta^1, ..., \beta^N)$, where

$$q_{\alpha\beta} = \begin{cases} \lambda_i, & \text{if } \beta - \alpha = e_i \\ \mu_i p_{i0}, & \text{if } \beta - \alpha = -e_i \\ \mu_i p_{ij}, & \text{if } \beta - \alpha = -e_i + e_j \end{cases} \qquad 1 \le i, j \le N$$

  Here e_i denote the vector $(0, ..., 0, 1, 0, ..., 0)$, having its i-th coordinate equal to 1.

- $P = (p_{ij})$ is a transition probability matrix on the state pace $\{0, 1, ..., N\}$. Assume $p_{ii} = p_{0i} = 0$, $p_{i0} > 0$, and $p_{00} = 1$, for all $1 \le i \le N$.

Open Jackson networks



Ergodicity of Jackson networks

- Jackson [1963, Management Science]:

$$\text{Ergodicity} \iff \nu_i < \mu_i \quad 1 \le i \le N$$

where ν_i satisfying $\nu_i = \lambda_i + \sum_{k=1}^{\infty} \sum_{j=1}^{N} \lambda_j p_{ji}^{(k)}$, here

$(p_{ij}^{(k)}) = P^k$, $P = (p_{ij})_{i,j=0,1,\ldots,N}$

Ergodicity of Jackson networks

- Jackson [1963, Management Science]:

$$\text{Ergodicity} \iff \nu_i < \mu_i \quad 1 \le i \le N$$

where ν_i satisfying $\nu_i = \lambda_i + \sum_{k=1}^{\infty}\sum_{j=1}^{N} \lambda_j p_{ji}^{(k)}$, here

$(p_{ij}^{(k)}) = P^k$, $P = (p_{ij})_{i,j=0,1,\ldots,N}$

- From Spieksma and Tweedie [1994,Commun.Statis. Stoch.Models.], we know the open Jackson network is exponentially ergodic, but no convergence rate was given.

Ergodicity of Jackson networks

- Symmetrizable condition:

$$\lambda_k p_{kl} p_{l0} = \lambda_l p_{lk} p_{k0} \qquad k \neq l, k, l = 1, 2, ..., N$$

Ergodicity of Jackson networks

- Symmetrizable condition:

$$\lambda_k p_{kl} p_{l0} = \lambda_l p_{lk} p_{k0} \qquad k \neq l, k, l = 1, 2, ..., N$$

- Under symmetrizable condition, we have $\nu_k = \frac{\lambda_k}{p_{k0}}$,

  Ergodicity $(\nu_k < \mu_k, 1 \leq k \leq N) \Longleftrightarrow \lambda_k < \mu_k p_{k0} \quad 1 \leq k \leq N$

2-node Jackson networks

- Theorem 3: For 2-node reversible Jackson network, we have

$$\frac{\overline{\lambda}_x}{q_x} \vee \frac{\overline{\lambda}_y}{q_y} \leq \mathrm{gap}(Q) \leq \overline{\lambda}_x \wedge \overline{\lambda}_y$$

where $1 \leq q_x, q_y < \infty$, $\overline{\lambda}_x = \left(\sqrt{\mu_1} - \sqrt{\lambda_1 + \frac{\lambda_2 p_{21}}{p_{20}}}\right)^2 > 0$, $\overline{\lambda}_y = \left(\sqrt{\mu_2} - \sqrt{\lambda_2 + \frac{\lambda_1 p_{12}}{p_{10}}}\right)^2 > 0$.

2-node Jackson networks

- Theorem 3: For 2-node reversible Jackson network, we have

$$\frac{\overline{\lambda}_x}{q_x} \vee \frac{\overline{\lambda}_y}{q_y} \le \mathrm{gap}(Q) \le \overline{\lambda}_x \wedge \overline{\lambda}_y$$

where $1 \le q_x, q_y < \infty$, $\overline{\lambda}_x = \left(\sqrt{\mu_1} - \sqrt{\lambda_1 + \frac{\lambda_2 p_{21}}{p_{20}}}\right)^2 > 0$,

$\overline{\lambda}_y = \left(\sqrt{\mu_2} - \sqrt{\lambda_2 + \frac{\lambda_1 p_{12}}{p_{10}}}\right)^2 > 0$.

- $\frac{\overline{\lambda}_x}{q_x} \vee \frac{\overline{\lambda}_y}{q_y} > 0$.

2-node Jackson networks

- Theorem 3: For 2-node reversible Jackson network, we have

$$\frac{\overline{\lambda}_x}{q_x} \vee \frac{\overline{\lambda}_y}{q_y} \le \text{gap}(Q) \le \overline{\lambda}_x \wedge \overline{\lambda}_y$$

where $1 \le q_x, q_y < \infty$, $\overline{\lambda}_x = \left(\sqrt{\mu_1} - \sqrt{\lambda_1 + \frac{\lambda_2 p_{21}}{p_{20}}}\right)^2 > 0$, $\overline{\lambda}_y = \left(\sqrt{\mu_2} - \sqrt{\lambda_2 + \frac{\lambda_1 p_{12}}{p_{10}}}\right)^2 > 0$.

- $\frac{\overline{\lambda}_x}{q_x} \vee \frac{\overline{\lambda}_y}{q_y} > 0$.
- In particular, if $p_{12} = p_{21} = 0$, we have

$$\text{gap}(Q) = (\sqrt{\mu_1} - \sqrt{\lambda_1})^2 \wedge (\sqrt{\mu_2} - \sqrt{\lambda_2})^2$$

N is finite or $N = \infty$

- For finite-node open Jackson network, we can also see it as a QBD, and its spectral gap is greater than 0.

N is finite or $N = \infty$

- For finite-node open Jackson network, we can also see it as a QBD, and its spectral gap is greater than 0.
- When $N = \infty$, it can be viewed as a reaction-diffusion process which arising from statistical physics. There are many many reaction-diffusion process examples in Chen's book [2004], but this new one is excluded. This may be the future work.

Acknowledgements

Thanks to my advisor Professor Mu-Fa CHEN
and thanks also to Professor Yong-Hua MAO
for their help and suggestions!

QED

Thank You !