On Switching Diffusions and Jump-diffusions

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(I) Switching Diffusions

Let $(X(t), Z(t))$ be a right continuously strong Markov process with the phase space $R^d \times N$, where $N := \{1, 2, \cdots, n_0\}$.

$$dX(t) = b(X(t), Z(t))dt + \sigma(X(t), Z(t))dB(t). \quad (1.1)$$

$$P\{Z(t+\Delta) = l | Z(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } k = l \end{cases} \quad (1.2)$$

uniformly in R^d ($\Delta \downarrow 0$), where $0 < q_{kl}(x) < +\infty$.

(I) Switching Diffusions

For the existence and uniqueness of the solution to (1.1) we make the following assumptions.

Assumption 1.1. For any $M > 0$, there is a constant $H_M > 0$ such that

$$|b(x,k) - b(y,k)| + |\sigma(x,k) - \sigma(y,k)| \le H_M|x-y|$$

and

$$|q_{kl}(x) - q_{kl}(y)| \le H_M|x-y|$$

for all $x, y \in U(M)$ and $k \ne l \in N$, where $U(M) := \{x \in R^d : |x| \le M\}$.

(I) Switching Diffusions

Define an operator L on $C^{2,1}(R^d \times R_+ \times N; R_+)$:

$$LV(x,t,k) = \frac{\partial}{\partial t}V(x,t,k) + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,k)\frac{\partial^2}{\partial x_i x_j}V(x,t,k) + \sum_{i=1}^{d} b_i(x,k)\frac{\partial}{\partial x_i}V(x,t,k) + \sum_{l\in N} q_{kl}(x)\big(V(x,t,l) - V(x,t,k)\big).$$

(I) Switching Diffusions

Assumption 1.2. There exists a nonnegative function $V(x, k)$ on $R^d \times N$ which is twice continuously differentiable in x such that for some constant $\alpha > 0$,

$$LV(x, k) \leq \alpha V(x, k), \quad (x, k) \in R^d \times N;$$

$$\inf\{V(x, k) : |x| \geq M\} \to \infty, \quad M \to \infty.$$

(I) Switching Diffusions

Under Assumptions 1.1 and 1.2, we can prove that stochastic differential equation (SDE) (1.1) has a unique continuously regular (i.e., non-explosive) solution $X(t)$. Throughout the rest of this work, we shall always, as standing hypotheses, assume that Assumptions 1.1 and 1.2 hold. Hence (1.1) and (1.2) together determine a unique regular solution $(X(t), Z(t))$ which is a strong Markov process.

(I) Switching Diffusions

Generally, the process $(X(t), Z(t))$ can be called a diffusion process with state-dependent switching. In particular, when the functions $q_{kl}(x)$ in (1.2) are independent of x (i.e., $q_{kl}(x) \equiv q_{kl} > 0$ for all $k \neq l$) and the second component $Z(t)$, which is independent of $B(t)$, is a Markov chain itself, the corresponding strong Markov process $(X(t), Z(t))$ then can be called a diffusion process with Markovian switching.

(I) Switching Diffusions

Switching diffusion processes can be used to describe a typical hybrid system that arises in many applications of systems with multiple modes or failure modes, such as fault-tolerant control systems, multiple target tracking, and flexible manufacturing systems:

- Ghosh, Arapostathis and Marcus (1993)
- Mariton (1990)

(I) Switching Diffusions

Theoretically, diffusion processes with state-dependent switching can be related to coupled elliptic PDE systems. When $\sigma(x,k) \equiv I$, large deviations and relevant results for them with some small numerical parameter were extensively studied:

- Eizenberg and Freidlin (1990,1993a,1993b)
- Xi (2005)

(I) Switching Diffusions

Stability for Diffusion processes with Markovian switching:

- Basak, Bisi and Ghosh (1996)
- Yuan and Mao (2003)
- Xi (2002, 2004)

Stability for Diffusion processes with state-dependent switching:

- Basak, Bisi and Ghosh (1999)

(I) Switching Diffusions

The f-exponential ergodicity for Diffusion processes with state-dependent switching:

- Xi and Zhao (2006)

The well known total variation norm is only a special case of the so-called f-norm. However, in the last reference, we only considered a particular case when the diffusion matrices are independent of k like $a(x)$.

(I) Switching Diffusions

The main reason for restricting us to the particular case is as follows. There we could not prove the Feller continuity for the general case of $(X(t), Z(t))$ when the diffusion matrices depend on not only x but also k by making use of the coupling methods. It is known that the Feller continuity of $(X(t), Z(t))$ is basic for its strong Feller continuity and f-exponential ergodicity results. Therefore, it is of considerable interest to study the problem on the Feller continuity for the general case of $(X(t), Z(t))$.

(I) Switching Diffusions

In this work we will introduce a more simply auxiliary process $(V(t), \psi(t))$ and make use of the Radon–Nikodym derivative of the measure induced by $(X(t), Z(t))$ in the space of trajectories with respect to the measure induced by $(V(t), \psi(t))$ to get the desired Feller continuity. On the basis of the Feller continuity, as we did before, we then prove the strong Feller continuity and further investigate the f-exponential ergodicity for the general diffusion processes with state-dependent switching.

(II) Feller Continuity: Special Case

Assumption 2.1. There is a constant $H > 0$ such that

$$|b(x,k)| \leq H, \quad |\sigma(x,k)| \leq H, \quad q_{kl}(x) \leq H,$$

$$|b(x,k) - b(y,k)| + |\sigma(x,k) - \sigma(y,k)| \leq H|x-y|$$

and

$$|q_{kl}(x) - q_{kl}(y)| \leq H|x-y|$$

for all $x, y \in R^d$ and $k \neq l \in N$.

(II) Feller Continuity: Special Case

We introduce an auxiliary process $(V(t), \psi(t))$:
$dV(t) = b(V(t), \psi(t))dt + \sigma(V(t), \psi(t))dB(t)$,
and let the

$$P\{\psi(t+\Delta) = l | \psi(t) = k\}$$
$$= \begin{cases} \Delta + o(\Delta), & \text{if } k \neq l, \\ 1 - (n_0 - 1)\Delta + o(\Delta), & \text{if } k = l \end{cases}$$

provided $\Delta \downarrow 0$, where n_0 is the number of elements in N.

(II) Feller Continuity: Special Case

We now introduce a metric $\lambda(\cdot,\cdot)$ on $R^d \times N$ as follows:

$$\lambda\big((x,m),(y,n)\big) = \rho(x,y) + d(m,n),$$

where

$$\rho(x,y) = |x-y|, \quad d(m,n) = \begin{cases} 0, & m = n, \\ 1, & m \neq n. \end{cases}$$

(II) Feller Continuity: Special Case

Lemma 2.2. Suppose that Assumption 2.1 holds. For all $T > 0$ and $\delta > 0$, we have that

$$P\left\{\max_{0\le t\le T} \lambda\Big((V^x(t), \psi^k(t)), (V^y(t), \psi^k(t))\Big) \ge \delta\right\} \to 0$$

as $\rho(x, y) \to 0$.

(II) Feller Continuity: Special Case

For a given $T > 0$, denote $\mu_1^T(\cdot)$ the measure induced by $(X(t), Z(t))$ and $\mu_2^T(\cdot)$ the measure induced by $(V(t), \psi(t))$ in the space of trajectories for $0 \le t \le T$, respectively. Then, $\mu_1^T(\cdot)$ is absolutely continuous with respect to $\mu_2^T(\cdot)$ and the corresponding Radon–Nikodym derivative has the following form.

(II) Feller Continuity: Special Case

$$M_T\big(V(\cdot),\psi(\cdot)\big) := \frac{d\mu_1^T}{d\mu_2^T}\big(V(\cdot),\psi(\cdot)\big)$$

$$= \prod_{i=0}^{n(T)-1} q_{\psi(\tau_i)\psi(\tau_{i+1})}\big(V(\tau_{i+1})\big)$$

$$\times \exp\Bigg(-\sum_{i=0}^{n(T)} \int_{\tau_i}^{\tau_{i+1}\wedge T} \big[q_{\psi(\tau_i)}(V(s)) - n_0 + 1\big] ds\Bigg)$$

(II) Feller Continuity: Special Case

where

$$q_k(x) = \sum_{l \neq k} q_{kl}(x),$$

τ_i is the sequence of Markov times defined by

$$\tau_0 = 0, \qquad \tau_{i+1} = \inf\{s > \tau_i : \psi(s) \neq \psi(\tau_i)\}$$

and

$$n(T) = \max\{i : \tau_i \leq T\}.$$

(II) Feller Continuity: Special Case

Lemma 2.3. Suppose that Assumption 2.1 holds. For all $T > 0$, we have that

$$E\left|M_T\big(V^x(\cdot),\psi^k(\cdot)\big) - M_T\big(V^y(\cdot),\psi^k(\cdot)\big)\right| \to 0$$

as $\rho(x,y) \to 0$.

Lemma 2.4. Suppose that Assumption 2.1 holds. For all $T > 0$ and $(x,k) \in R^d \times N$, the Radon–Nikodym derivative $M_T\big(V^x(\cdot),\psi^k(\cdot)\big)$ is integrable.

(II) Feller Continuity: Special Case

Using Lemmas 2.2, 2.3 and 2.4., we can prove

Theorem 2.6. Suppose that Assumption 2.1 holds. The process $(X(t), Z(t))$ determined by (1.1) and (1.2) has Feller property.

Proof:

$$Ef(X^x(t), Z^k(t)) = Ef(V^x(t), \psi^k(t)) \cdot M_t\big(V^x(\cdot), \psi^k(\cdot)\big)$$

- Wu (2001)

(III) Feller Continuity: General Case

For any given $M > 0$, set

$$\tau_M = \inf\{t \geq 0 : (X^x(t), Z^k(t)) \notin U(M) \times N\},$$

where $U(M) = \{x \in R^d : |x| \leq M\}$.

Lemma 3.1. For any fixed bounded domain D in R^d and $t \geq 0$, we then have $P\{\tau_M \leq t\} \to 0$ uniformly over (x, k) in $D \times N$ as $M \uparrow \infty$.

Using a truncation argument, we then can prove the Feller continuity for the general case of $(X(t), Z(t))$.

(III) Feller Continuity: General Case

Theorem 3.2. Under Assumptions 1.1 and 1.2 (the standing hypotheses) but not Assumption 2.1, The process $(X(t), Z(t))$ determined by (1.1) and (1.2) still has Feller property.

Remark 3.3. The dependence on k of the diffusion matrices does perplex the problem greatly. Actually, even if the diffusion matrices are independent of x and take the simple forms like $a(k)$, $k \in N$, we still could not prove the Feller continuity result by using the coupling methods.

(IV) Strong Feller Continuity

Assumption 4.1. There exists a positive constant H such that

$$\sup\{q_{kl}(x) : x \in R^d, k \neq l \in N\} \leq H < +\infty.$$

Let $\{\eta_m\}$ be the sequence of Markov times defined by

$$\eta_0 = 0, \qquad \eta_m = \inf\{s > \eta_{m-1} : Z(t) \neq Z(\eta_{m-1})\}.$$

Set $\eta = \lim_{m\to\infty} \eta_m$ and $J(t) = \max\{m : \eta_m \leq t\}$.

(IV) Strong Feller Continuity

Proposition 4.2. Suppose that Assumption 4.1 holds.

$$P^{(x,k)}\big(\eta = +\infty\big) = 1, \qquad P^{(x,k)}\big(J(t) < +\infty\big) = 1;$$

$$\begin{aligned} &P^{(x,k)}\big((X(t), Z(t)) \in D\big) \\ &= \textstyle\sum_{m=1}^{\infty} P^{(x,k)}\big((X(t), Z(t)) \in D,\ J(t) = m\big). \end{aligned}$$

For any given $t > 0$, set

$$P(t) = \{P(t,(x,k),A) : (x,k) \in R^d \times N, A \in \mathcal{B}(R^d \times N)\}$$

(IV) Strong Feller Continuity

For each $k \in N$, let $X_k(t)$ satisfy the following SDE in R^d,

$$dX_k(t) = b(X_k(t), k)dt + \sigma(X_k(t), k)dB(t).$$

Assumption 4.3. For each $k \in N$, assume that the single diffusion process $X_k(t)$ is a strong Feller process and it has transition density $p^{(k)}(t, x, y)$ with respect to the Lebesgue measure.

(IV) Strong Feller Continuity

Let the reference measure $\mu(\cdot)$ is the product measure on $R^d \times N$ of the Lebesgue measure on R^d and the counting measure on N.

Theorem 4.4. Suppose that Assumptions 4.1 and 4.3 hold. For every $(x, k) \in R^d \times N$ and every $t > 0$, the transition probability $P(t, (x, k), \cdot)$ of $(X(t), Z(t))$ is absolutely continuous with respect to $\mu(\cdot)$. Furthermore, for any given $t > 0$, the transition probability kernel $P(t)$ defined as before is strong Feller continuous.

(V) Exponential Ergodicity

In this section we investigate the exponential ergodicity for the strong Markov process $(X(t), Z(t))$. For this we make an assumption.

Assumption 5.2. For each $k \in N$, assume that the single diffusion process $X_k(t)$ determined before is a strong Feller process, that it has transition density $p^{(k)}(t, x, y)$ with respect to the Lebesgue measure, and that the support of $X_k(t)$ is equal to R^d for all $t > 0$.

(V) Exponential Ergodicity

For any positive function $f(x,k) \geq 1$ defined on $R^d \times N$ and any signed measure $\nu(\cdot)$ defined on $\mathcal{B}(R^d \times N)$ we write

$$\|\nu\|_f = \sup\{|\nu(g)| : |g| \leq f\}.$$

Note that the well known total variation norm $\|\nu\|$ is just $\|\nu\|_f$ in the special case where $f \equiv 1$.
Hence, the f-norm is a very strong norm.
Moreover, the larger f is, the stronger f-norm will be.

(V) Exponential Ergodicity

For a function $\infty > f \geq 1$ on $R^d \times N$, Markov process $(X(t), Z(t))$ is said to be f-exponentially ergodic if there exist a probability measure $\pi(\cdot)$, a constant $\theta < 1$ and a finite-valued function $\Theta(x, k)$ such that

$$\|P(t, (x, k), \cdot) - \pi(\cdot)\|_f \leq \Theta(x, k)\theta^t$$

for all $t \geq 0$ and all $(x, k) \in R^d \times N$.

(V) Exponential Ergodicity

Moreover, a nonnegative function $V(x,k)$ defined on $R^d \times N$ is called a norm-like function if $V(x,k) \to \infty$ as $|x| \to \infty$ for all $k \in N$. Now we need to introduce a Foster-Lyapunov drift condition as follows. For some α, $\beta > 0$ and a norm-like function $V(x,k)$ which is twice continuously differentiable in x,

$$LV(x,k) \le -\alpha V(x,k) + \beta \tag{5.5}$$

for $(x,k) \in R^d \times N$.

(V) Exponential Ergodicity

Theorem 5.3. Suppose that (5.5) and Assumptions 3.1 and 5.2 hold. Then the strong Markov process $(X(t), Z(t))$ is f-exponentially ergodic with $f = V + 1$.

Remark. *One may further consider the f-exponential ergodicity for jump-diffusions with Markovian switching or state-dependent switching.*