Harnack Inequalities for Diffusion Semigroups and Applications

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Contents

Elliptic Harnack Inequalities

For any open domain $D \subset \mathbb{R}^d$ and any compact $K \subset D$, there exists a constant $C = C(K, D)$ such that

$$\sup_K u \le C \inf_K u$$

holds for all no-negative harmonic functions on D.
(Harnack, 1880s, $d = 2$)

Extension: the same holds for solutions to $Lu = 0$ for an elliptic second differential operator L with smooth coefficients.

Global version: there exists a continuous positive function C on $\mathbb{R}^d \times \mathbb{R}^d$, such that for any non-negative function u with $Lu = 0$ on $\mathbb{R}^d$,

$$u(x) \le C(x, y)u(y), \quad x, y \in \mathbb{R}^d.$$

Parabolic Harnack Inequalities

Consider the heat equation ($f \in C_b(\mathbb{R}^d)$)

$$\partial_t u = Lu, \quad u(0,\cdot) = f \geq 0.$$

Minimal solution: $u(t,x) = P_t f(x) := \mathbb{E}^x f(X_t) 1_{\{t<\xi\}}$,
X_t is the L-diffusion process with life time ξ.

Question: is there $C : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that

$$P_t f(x) \leq C(t,x,y) P_t f(y), \quad x,y \in \mathbb{R}^d, f \geq 0?$$

No in general: wrong even for the classical heat equation on $\mathbb{R}^d$ where $L = \Delta$.

True for very exceptional situation (Wang,JFA 2006).

Modified Parabolic Harnack Inequalities

By Shifting times **(Moser, Trudinger 1960s)**: for elliptic L with continuous coefficients, there exists $C : (0,\infty)^2 \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that

$$P_t f(x) \le C(t,s,x,y) P_{t+s} f(y), \quad s,t > 0, x,y \in \mathbb{R}^d$$

holds for all bounded $f \ge 0$.

Explicit Inequality **(Li-Yau 1986)**: Let $L = \Delta$ on a completed Riemannian manifold M with Ricci curvature bounded below by $-K$, for some constant $K \ge 0$ ($K = 0$ on $\mathbb{R}^d$!),

$$P_t f(x) \le \big(P_{t+s} f(y)\big) \Big(\frac{t+s}{s}\Big)^{d\alpha/2} \exp\Big[\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha d K s}{4(\alpha-1)}\Big]$$

for all $x,y \in M, s,t > 0, \alpha > 1$.

Extensions: **Bakry-Qian(1999); X.D. Li(2005)** for $L = \Delta + Z$ with curvature-dimension conditions.

Applications: heat kernel, log-Sobolev/Poincaré inequalities....

Dimension-Free Harnack Inequalities

Fact: the above mentioned Harnack inequalities essentially depend on the dimension. To study dimension-free properties of diffusion processes (e.g. exponential convergence rate, Poincaré or log-Sobolev inequalities...), we introduce a dimension-free version of Harnack inequality by shifting a power.

Dimension-free Harnack Inequality

(Wang, PTRF 1997; Int. Equat. Operat. Th. 04)

Let $L = \Delta + Z$ on M. Ricci $-\nabla Z \geq -K$ if and only if

$$(P_t f)^\alpha(x) \leq \big(P_t f^\alpha(y)\big) \exp\Big[\frac{K\alpha\rho(x,y)^2}{2(\alpha-1)(1-\exp[-2Kt])}\Big]$$

holds for all $x, y \in M, t > 0, \alpha > 1$.

Dimension-Free Harnack Inequalities

Applications: functional inequalities, estimates on semigroups and heat kernels, transportation-cost inequalities... for both finite/infinite dimensions.

Key Point of the Proof: Ricci $-\nabla Z \geq -K$ is equivalent to the gradient estimate

$$|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|, \quad t \geq 0, f \in C_b(M).$$

So, the regularity of the manifold and the drift Z is required.

Next Part: Introduce a coupling method for singular situations.

Coupling Method: a general idea

Let P_t be the semigroup of a strong Markov process on some Polish space E.

For any $x \neq y, t > 0$ and $\alpha > 1$, we aim to find a constant $C(t, x, y) > 0$ such that

$$(P_t f(x))^\alpha \leq (P_t f^\alpha)(y) C(t, x, y), \quad f \geq 0.$$

Idea: Let (X_s, Y_s) be a coupling of the process starting at (x, y). If

$$\tau := \inf\{s \geq 0 : X_s = Y_s\} \leq t, \text{a.s.}$$

By the strong Markov property we may let $X_s = Y_s$ for $s \geq \tau$. So, if $\tau \leq t$ one has $X_t = Y_t$ so that

$$P_t f(x) = \mathbb{E} f(X_t) = \mathbb{E} f(Y_t) = P_t f(y).$$

Too strong to be true!

Coupling Method: general idea

Modification: For X_s the Markov process starting from x, construct another process Y_s with $Y_0 = y$ such that

(1) $X_t = Y_t$ a.s.
(2) $P_s f(y) = \mathbb{E}[Rf(Y_s)]$ for a probability density R.

Note: (1) can be realized by adding a strong enough drift; while (2) can be confirmed by a Girsanov type theorem.

$$\begin{aligned}(P_t f)^\alpha(y) &= (\mathbb{E}[Rf(Y_t)])^\alpha = (\mathbb{E}[Rf(X_t)])^\alpha \\ &\le (P_t f^\alpha(x))(\mathbb{E}R^{\frac{\alpha}{\alpha-1}})^{\alpha-1}.\end{aligned}$$

This implies the desired Harnack inequality provided

(3) $\mathbb{E}R^p < \infty, \quad p > 1.$

Examples

Example 1. $L = \Delta + Z$ on $\mathbb{R}^d$. Coupling by solving SDEs

$$\mathrm{d}X_s = \sqrt{2}\mathrm{d}B_s + Z(X_s)\mathrm{d}s, \quad X_0 = x;$$
$$\mathrm{d}Y_s = \sqrt{2}\mathrm{d}B_s + \Big\{Z(Y_s) + \xi_s \frac{X_s - Y_s}{|X_s - Y_s|} 1_{\{s<\tau\}}\Big\}\mathrm{d}s, \quad Y_0 = y.$$

Then

$$\mathrm{d}|X_s - Y_s| \le \Big(\frac{\langle Z(Y_s) - Z(X_s), X_s - Y_s\rangle}{|X_s - Y_s|} - \xi_s\Big)\mathrm{d}s$$

for $s < \tau$. Taking ξ_s such that

$$\xi_s \ge \frac{\langle Z(Y_s) - Z(X_s), X_s - Y_s\rangle}{|X_s - Y_s|} + \frac{|x-y|}{t}, \quad s \le \tau,$$

we obtain $\tau \le t$. By Girsanov theorem, the above argument applies to
$R := \exp\big[\sqrt{2}\int_0^t \xi_s \mathrm{d}b_s - \int_0^t \xi_s^2 \mathrm{d}s\big]$.

Examples

Example 2. $L = \Delta + Z$ on a Riemannian manifold with curvature unbounded below, contains diffusions with non-constant diffusion coefficients on $\mathbb{R}^d$.
Arnaudon/Thalmaier/Wang, Bull. Sci. Math. 2006

Example 3. Dirichlet semigroup P_t^D on a bounded domain D: coupling successful before t and the hitting times to the boundary. Technical difficulty: unbounded stopping times are involved, so that the corresponding process R is not L^p-integrable for $p > 1$. By estimating moments of $\lim_{\rho(x,y)\to 0} \frac{|1-R|}{\rho(x,y)}$, we obtain gradient estimate

$$|\nabla P_t^D f(x)| \le \delta P_t^D \Big\{ f \log \frac{f}{P_t^D f(x)} \Big\}(x) + C(\delta, x) P_t^D f(x)$$

for all $\delta > 0, x \in D$, which implies the desired Harnack inequality.
Arnaudon/Thalmaier/Wang, preprint 2007

Examples

Example 4. Stochastic porous medium equation:
$(E, \mathcal{F}, \mu)$ a σ-finite measure space,
$(L, \mathcal{D}(L))$ a negative definite self-adjoint operator with $0 \notin \sigma(L)$.
Let $r \geq 1$ and $Q \geq c\mathrm{Id}$ a linear operator on $L^2(\mu)$.
Let W_t be the cylindrical Brownian motion on $L^2(\mu)$.

If the spectrum of L is discrete enough, then the semigroup of the solution to

$$\mathrm{d}X_t = (LX_t^r)\mathrm{d}t + Q\mathrm{d}W_t$$

satisfies the Harnack inequality, and is strong Feller.

Wang, Ann. of Probab. 2007.

Application: Contractivity Properties

Let P_t be the diffusion semigroup generated by $L := \Delta + \nabla V$ on a complete connected Riemannian manifold M, with $\mathrm{d}\mu := \mathrm{e}^{V(x)}\mathrm{d}x$ a probability measure.

Definition **(Nelson JFA 1973; Davies/Simons JFA 1984)**

(a) Hypercontractivity: $\|P_t\|_{L^2(\mu)\to L^4(\mu)} \leq 1$ for some $t > 0$;

(b) Supercontractivity: $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ for all $t > 0$;

(c) Ultracontractivity: $\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} < \infty$ for all $t > 0$.

Application: Contractivity Properties

Theorem

(Nelson JFA 1973; Davies/Simons JFA 1984)

(1) *Hypercontractivity* $\Leftrightarrow$ *there exists* $C > 0$ *such that*

$$\mu(f^2 \log f^2) \le C\mu(|\nabla f|^2), \quad \mu(f^2) = 1.$$

(2) *Supercontractivity* $\Leftrightarrow$ *there exists* $\beta : (0,\infty) \to (0,\infty)$ *such that*

$$\mu(f^2 \log f^2) \le r\mu(|\nabla f|^2) + \beta(r), \quad r > 0, \mu(f^2) = 1.$$

(3) *Ultracontractivity* $\Leftrightarrow$ P_t *has transition density* $p_t(x,y)$ *w.r.t.* μ *which is bounded in* (x,y)*:*

$$\|P_t\|_{L^1(\mu)\to L^\infty(\mu)} = \sup_{x,y} p_t(x,y), \quad t > 0.$$

Application: Contractivity Properties

Starting from the Harnack inequality

$$(P_t f(x))^\alpha \le C(t,x,y) P_t f^\alpha(y),$$

we have for $f \ge 0$ with $\mu(f^\alpha) = 1$,

$$(P_t f(x))^\alpha \int_M C(t,x,y)^{-1} \mu(\mathrm{d}y) = \int_M P_t f^\alpha(y) \mu(\mathrm{d}y) = 1.$$

Thus,

$$P_t f(x) \le \frac{1}{\left(\int_M C(t,x,y)^{-1}\mu(\mathrm{d}y)\right)^{1/\alpha}} =: \Psi_t(x).$$

Therefore, for any $p > 1$, $\|P_t\|_{L^\alpha(\mu)\to L^{p\alpha}(\mu)} < \infty$ provided $\Psi_t \in L^p(\mu)$.

Combining this fact with the explicit Harnack inequality we introduced before, we obtain

Application: Contractivity Properties

Let ρ be the Riemannian distance function to a fixed point in M.

Theorem

(Wang 01; Röckner/Wang 03) *Assume* $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ *for some* $K \geq 0$.

(1) P_t *is Hypercontractive* $\Leftarrow$ *there exists* $\lambda > \frac{K}{2}$ *such that* $\mu(\mathrm{e}^{\lambda\rho^2}) < \infty$.

(2) P_t *is Supercontractive* $\Leftrightarrow \mu(\mathrm{e}^{\lambda\rho^2}) < \infty$ *for all* $\lambda > 0$.

(3) P_t *is Ultracontractive* $\Leftrightarrow \|P_t \mathrm{e}^{\lambda\rho^2}\|_\infty < \infty$ *for any* $t, \lambda > 0$.

Remark. The sufficient condition in (1) is optimal: for any $K > 0$ and $\lambda < \frac{K}{2}$, there exists example such that $\mathrm{Ric} - \mathrm{Hess}_V \geq -K$ and $\mu(\mathrm{e}^{\lambda\rho^2}) < \infty$ but P_t is not hypercontractive.

Application: Contractivity Properties

Example

Let the Ricci curvature be bounded from below, and let $Z = -\nabla\rho^\delta \log^\alpha(1+\rho)$ *for some* $\delta, \alpha \geq 0$.

(1) P_t *is hypercontractive if and only if* $\delta \geq 2$.
(*well-known for* $M = \mathbb{R}^d$ *and* $\alpha = 0$ *by Nelson and Gross*).

(2) P_t *is supercontractive if and only if either* $\delta > 2$ *or* $\delta = 2$ *but* $\alpha > 0$.

(3) P_t *is ultracontractive if and and only if either* $\delta > 2$ *or* $\delta = 2$ *but* $\alpha > 1$.

Application: Heat Kernel Estimates

Starting from the Harnack inequality

$$(P_t f(x))^2 \le C(t,x,y) P_t f^2(y),$$

we have for $f \ge 0$ with $\mu(f^2) = 1$, we have

$$P_t f(x) \le \frac{1}{\big(\int_M C(t,x,y)^{-1}\mu(\mathrm{d}y)\big)^{1/2}}.$$

Let $p_t(x,y)$ be the density of P_t w.r.t. μ. Applying this inequality to $f := p_t(x,\cdot)/p_{2t}(x,x)^{1/2}$, we obtain

$$p_{2t}(x,x) \le \frac{1}{\int_M C(t,x,y)^{-1}\mu(\mathrm{d}y)}.$$

Combining this with an argument of Grigoy'an (1997), we obtain Gaussian type point wise upper bound on $p_t(x,)$

Application: Heat Kernel Estimates

For instance:

Theorem

(Gong/Wang 01) If $\mathrm{Ric} - \mathrm{Hess}_V$ *is bounded below, then for any* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that*

$$p_t(x,y) \le \frac{C(\varepsilon)\exp[-\frac{\rho(x,y)^2}{(4+\varepsilon)t}]}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}, \quad x,y \in M, t > 0.$$

Sharp for short time.

Other Applications

Log-Sobolev constant: Wang Ann. Probab. 99; J. Operat. Th. 01.

Asymptotic behavior of transition probability for ∞-dimension diffusions: Aida/Kawabi, Stochastic Analysis and Related Topics 98; Aida/Zhang Pot. Anal. 02; Kawabi Pot. Anal. 05

Transportation cost inequalities: Bobkov/Gentil/Ledoux J. Math. Pures Appl. 01

Regularity for SPDEs: Wang Ann. Probab. 07

Thank You !