Boundary Harnack Principle for Subordinate BMs

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We will show that the boundary Harnack principle is valid for a large class of subordinate BM.

Classical BHP

The BHP for classical harmonic functions was proved independently in the late 1970's by, Ancona, Dahlberg and Wu. It is a very deep result in harmonic analysis and has very important applications. Roughly speaking the classical BHP can be stated as follows.

Classical BHP Suppose D is a bounded Lipschitz domain in $\mathbb{R}^d$, V is an open set, and K is a compact subset of V. If u and v are positive harminic functions in D which vanish continuously in $V \cap \partial D$, then u/v is bounded between two positive numbers in $D \cap K$.

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Very recently Bogdan, Kulczycki and Kwasnicki proved a version of the BHP for isotropic stable processes on arbitrary open sets.

By using some perturbation methods, the BHP has been generalized to some classes of rotationally invariant Lèvy processes including the relativistic stable processes and truncated stable processes. These processes can be regarded as perturbations of the isotropic stable processes and their Green functions on bounded smooth domains are comparable to their counterparts for isotropic stable processes.

See Chen-Song, Ryznar, Grzwny-Ryznar and Kim-Song1, Kim-Song2 for results on the comparison of Green functions. The comparison of Green functions played a crucial role in proving the BHP.

We will show that, under minimal conditions, the BHP is valid for subordinate BMs with Lévy exponents of the form $\Phi(\xi) = |\xi|^\alpha \ell(|\xi|^2)$ for some $\alpha \in (0, 2)$ and some positive function ℓ which is slowly varying at ∞. Examples of this class of subordinate BMs include, among others, relativistic stable processes and mixtures of isotropic stable processes.

The Green functions of subordinate BMs considered here behave like $\frac{c}{|x|^{d-\alpha}\ell(|x|^{-2})}$ near the origin. So these subordinate BMs can not be regarded as perturbations of isotropic stable processes in general.

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Subordinate BMs have been used in mathematical finance to model stock prices by Madan, Yor and their collaborators. Understanding more general subordinate BMs provides more tools for modeling.

One of the key ingredients is a sharp upper bound for expected exit time from a ball which, in the case of stable processes, follows easily from the explicit formula for the Green function of a ball. However, in the present case, the desired upper bound is pretty difficult to establish. We have to use the fluctuation theory for real-valued Lévy processes to accomplish this.

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In this talk, I will concentrate on getting the desired sharp upper bound for expected exit time from balls.

1-d subordinate BM

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Suppose that $W = (W_t : t \geq 0)$ is a 1-d BM and that $S = (S_t : t \geq 0)$ is a subordinator independent of W. We assume that the Laplace exponent of S is a complete Bernstein function of the form

$$\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$$

for some $\alpha \in (0, 2)$ and some positive function ℓ which is slowly varying at ∞.

Bernstein Functions

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A function $\phi : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation given by

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A function ϕ is the Laplace exponent of a subordinator iff it is a BF with $\lim_{\lambda\to 0} \phi(\lambda) = 0$.

Special Bernstein Functions

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A Bernstein function is special iff the restriction of the potential measure of the corresponding subordinator to $(0, \infty)$ has a decreasing density.

Complete Bernstein Functions

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A CBF is a SBF. The restriction, to $(0, \infty)$, of the potential measure of a subordinator whose LE is a CBF has a completely monotone density. Most of the explicit Bernstein functions we know are complete Bernstein functions.

The potential measure of S has a decreasing density u satisfying

$$(0) \qquad u(t) \sim \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} \frac{1}{\ell(t^{-1})}, \qquad t \to 0.$$

and its Lévy measure has a density $\mu(t)$ satisfying

$$(1) \qquad \mu(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)} \frac{\ell(t^{-1})}{t^{1+\alpha/2}}, \qquad t \to 0.$$

The subordinate BM $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is an isotropic Lévy process with Lévy exponent

$$\Phi(\theta) = \phi(\theta^2) = |\theta|^\alpha \ell(\theta^2), \qquad \forall \theta \in \mathbb{R}.$$

Let χ be the Laplace exponent of the ladder height process of X. It is known that that

(2)
$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\Phi(\lambda\theta))}{1+\theta^2}d\theta\right)$$
$$= \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(|\theta|^\alpha\lambda^\alpha\ell(\theta^2\lambda^2))}{1+\theta^2}d\theta\right)$$

for all $\lambda > 0$.

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Proposition Under some natural conditions on ℓ, we have

$$\lim_{\lambda\to\infty} \frac{\chi(\lambda)}{\lambda^{\alpha/2}(\ell(\lambda^2))^{1/2}} = 1. \tag{5}$$

Examples

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for some $\alpha \in (0, 2)$. ϕ is a CBF which can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = \frac{(\lambda + 1)^{\alpha/2} - 1}{\lambda^{\alpha/2}}.$$

ℓ satisfies the desired natural condition.

Suppose $0 < \beta < \alpha < 2$ and define

$$\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\beta/2}.$$

Then ϕ is a CBF which can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

$$\ell(\lambda) = 1 + \lambda^{(\beta-\alpha)/2}.$$

ℓ satisfies the desired natural condition.

Suppose that $\alpha \in (0, 2)$ and $\beta \in (0, 2 - \alpha)$.

$$\phi(\lambda) = \lambda^{\alpha/2}(\log(1 + \lambda))^{\beta/2}.$$

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Then ϕ is a complete Bernstein function which can be written as $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ with

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ℓ satisfies the desired natural condition.

It follows from the Propositions above that the potential measure V of the ladder height process of X has a decreasing density v. Using the assumed asymptotic behavior of χ, one can easily apply Karamata's Tauberian theorem and the monotone density theorem to get the following result.

Proposition As $x \to 0$, we have

$$V((0,x)) \sim \frac{x^{\alpha/2}}{\Gamma(1+\alpha/2)(\ell(x^{-2}))^{1/2}},$$

$$v(x) \sim \frac{x^{\alpha/2-1}}{\Gamma(\alpha/2)(\ell(x^{-2}))^{1/2}}.$$

Let $G^{(0,\infty)}(x,y)$ be the Green function of $X^{(0,\infty)}$, the process obtained by killing X upon exiting from $(0,\infty)$. Then

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Proposition For any $x, y > 0$ we have

$$G^{(0,\infty)}(x,y) = \begin{cases} \int_0^x v(z)v(y+z-x)dz, & x \leq y, \\ \int_{x-y}^x v(z)v(y+z-x)dz, & x > y. \end{cases}$$

For any $r > 0$, let $G^{(0,r)}$ be the Green function of $X^{(0,r)}$, the process obtained by killing X upon exiting from $(0, r)$. Then we have the following result.

Theorem For any $R > 0$, there exists $C = C(R) > 0$ such that

$$\int_0^r G^{(0,r)}(x,y)dy \le C \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{x^{\alpha/2}}{(\ell(x^{-2}))^{1/2}}$$

for all $x \in (0,r)$ and $r \in (0,R)$.

As a consequence, we have

Theorem For any $R > 0$, there exists $C = C(R) > 0$ such that

$$\int_0^r G^{(0,r)}(x,y)dy \leq C\frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}}\left(\frac{x^{\alpha/2}}{(\ell(x^{-2}))^{1/2}} \wedge \frac{(r-x)^{\alpha/2}}{(\ell((r-x)^{-2}))^{1/2}}\right)$$

for all $x \in (0,r),\ r \in (0,R)$.

High dimensional subordinate BM

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$d \geq 2$, $\alpha \in (0, 2)$. $B = (B_t : t \geq 0)$ is a BM on $\mathbb{R}^d$. $S = (S_t : t \geq 0)$ is a subordinator independent of B and that its Laplace exponent ϕ is a CBF satisfying all the assumptions above. That is, there is a positive function ℓ on $(0, \infty)$ which is slowly varying at ∞ such that $\phi(\lambda) = \lambda^{\alpha/2}\ell(\lambda)$ for all $\lambda > 0$ and that ℓ satisfied the natural condition we referred to.

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$u(t)$ and $\mu(t)$ are the potential density and Lévy density of S respectively.

$X = (X_t : t \geq 0)$ is the subordinate BM defined by $X_t = B_{S_t}$. The Green function G of X is given by the following formula

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u(t) dt, \qquad x \in \mathbb{R}^d.$$

Let J be the jumping function of X, then

$$J(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} \mu(t) dt, \qquad x \in \mathbb{R}^d.$$

Thus $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt, \qquad r > 0.$$

Theorem Under some additional natural condition on ℓ, the Green function G of X satisfies

$$G(x) \sim \frac{\alpha\Gamma((d-\alpha)/2)}{2^{\alpha+1}\pi^{d/2}\Gamma(1+\alpha/2)} \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}$$

as $|x| \to 0$.

Theorem Under some additional natural condition on ℓ, the function j satisfies the following

$$j(r) \sim \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\alpha/2)}\frac{\ell(r^{-2})}{r^{d+\alpha}}$$

as $r \to 0$.

A nonnegative function u is said to be harmonic wrt to X in an open set $\subset \mathbb{R}^d$ if for every open set B with $\overline{B} \subset D$,

$$u(x) = \mathbb{E}_x u(X_{\tau_B}), \qquad x \in B.$$

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u is said to be regular harmonic in D wrt X if

$$u(x) = \mathbb{E}_x u(X_{\tau_D}), \qquad x \in B.$$

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Theorem There exist $r_1 \in (0,1)$ and $C > 0$ such that for every $r \in (0, r_1)$, every $x_0 \in \mathbb{R}^d$, and every nonnegative function f on $\mathbb{R}^d$ which is harmonic in $B(x_0, r)$ with respect to X, we have

$$\sup_{y \in B(x_0, r/2)} f(y) \le C \inf_{y \in B(x_0, r/2)} f(y).$$

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Proposition For any $R>0$, there exists $C=C(R)>0$ such that for every open subset D with $\operatorname{diam}(D)\le R$,

$$G_D(x,y)\ \le\ C\,\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}}. \tag{7}$$

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Lemma For any $R>0$, there exists $C=C(R)>0$ such that for every $r\in(0,R)$ and $x_0\in\mathbb{R}^d$,

$$\mathbb{E}_x[\tau_{B(x_0,r)}]\ \le\ C\,\frac{r^{\alpha/2}}{(\ell(r^{-2}))^{\frac12}}\frac{(r-|x-x_0|)^{\alpha/2}}{(\ell((r-|x-x_0|)^{-2}))^{\frac12}}$$

for all $x\in B(x_0,r)$.

Proof Without loss of generality, we may assume that $x_0 = 0$. For $x \neq 0$, put $Z_t = \frac{X_t \cdot x}{|x|}$. Then Z_t is a Lévy process on $\mathbb{R}$ with

$$\mathbb{E}(e^{i\theta Z_t}) = \mathbb{E}(e^{i\theta \frac{x}{|x|} \cdot X_t}) = e^{-t|\theta|^\alpha \ell(\theta^2)}, \qquad \theta \in \mathbb{R}.$$

Thus Z_t is the type of one-dimensional subordinate BM we studied before.

It is easy to see that, if $X_t \in B(0,r)$, then $|Z_t| < r$, hence

$$\mathbb{E}_x[\tau_{B(0,r)}] \le \mathbb{E}_{|x|}[\tilde{\tau}],$$

where $\tilde{\tau} = \inf\{t > 0 : |Z_t| \ge r\}$. Now the desired conclusion follows easily from our result on 1-d subordinate BM.

We also have

Lemma There exist $r_2 \in (0, r_1]$ and $C > 0$ such that for every positive $r \le r_2$ and $x_0 \in \mathbb{R}^d$,

$$\mathbb{E}_{x_0}\big[\tau_{B(x_0,r)}\big] \;\ge\; C\,\frac{r^{\alpha}}{\ell\big(r^{-2}\big)}.$$

Define

$$K_D(x,y) := \int_D G_D(x,z)J(z-y)dz \tag{8}$$

on $D \times \overline{D}^c$. Then we have

$$\mathbb{E}_x\left[f(X_{\tau_D});\ X_{\tau_D-} \neq X_{\tau_D}\right] = \int_{\overline{D}^c} K_D(x,y)f(y)dy.$$

Proposition There exist $C_1, C_2 > 0$ such that for every $r \in (0, r_2)$ and $x_0 \in \mathbb{R}^d$,

$$K_{B(x_0,r)}(x,y) \le C_1\, j(|y - x_0| - r) \cdot \frac{r^{\alpha/2}}{(\ell(r^{-2}))^{1/2}} \frac{(r - |x - x_0|)^{\alpha/2}}{(\ell((r - |x - x_0|)^{-2}))^{1/2}}, \tag{9}$$

and

$$K_{B(x_0,r)}(x_0,y) \ \ge \ C_2\, J(2(y - x_0)) \frac{r^{\alpha}}{\ell(r^{-2})}. \tag{10}$$

Proposition For every $a \in (0,1)$, there exists $C = C(a) > 0$ such that for every $r \in (0, r_2)$, $x_0 \in \mathbb{R}^d$ and $x_1, x_2 \in B(x_0, ar)$,

$$K_{B(x_0,r)}(x_1, y) \le C K_{B(x_0,r)}(x_2, y), \qquad y \in \overline{B(x_0, r)}^c.$$

Proposition For every $a \in (0,1)$, there exists $C = C(a) > 0$ such that for every $r \in (0, r_4]$ and $x_0 \in \mathbb{R}^d$,

$$K_{B(x_0,r)}(x,y) \;\leq\; C\,\frac{r^{\alpha/2-d}}{(\ell(r^{-2}))^{1/2}}\frac{(\ell(((|y-x_0|-r)^{-2}))^{1/2}}{(|y-x_0|-r)^{\alpha/2}}$$

for all $x \in B(x_0, ar)$, $y \in \{r < |x_0 - y| \leq 2r\}$.

BHP Suppose D is a bdd κ-fat open set. There exist $r_5 := r_5(D, \alpha)$ and $C = C(D, M) > 1$ such that if $2r \le r_5$ and $Q \in \partial D$, then for any nonneg fcns u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ wrt X and vanish in $D^c \cap B(Q, 2r) \cup B(Q, M)^c$, we have

$$C^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le C \frac{u(A_r(Q))}{v(A_r(Q))}$$

for all $x \in D \cap B(Q, \frac{r}{2})$.