On the Convergence Rates to the Equilibrium for the Brownian Motion with Divergence Free Drifts

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1. Introduction

$B(t)$ is the d-dim Brownian motion and $X^{(c)}$ satisfies

$$dX^{(c)}(t) = cb(X^{(c)}(t))dt + dB(t). \tag{1.1}$$

$b(\cdot)$ is a smooth vector field with period 1.

b is divergence free. That is,

$$div(b) = 0. \tag{1.2}$$

c is a large parameter.

$X^{(c)}(t)$ is a diffusion process on d-dim torus.

$X^{(c)}(t)$ has Lebsegue measure (on torus) as the invariance measure (by (1.2)).

This is a particular example of a more general class of diffusion processes,

$$dX(t) = (-\nabla U(X(t)) + b(X(t)))dt + dB(t),$$

with U, b periodic and satisfying

$$div(b \exp(-2U)) = 0,$$

such that they have μ as the invariance measure,

$$d\mu = \frac{1}{Z}\exp(-2U(x))dx.$$

Such diffusion processes appear in MCMC(Markov Chain Monte Carlo),

One chooses particular b to simulate μ.

We may also consider μ on R^d and we do not assume periodicity of U.

A main concern is how well the distribution of $X(t)$ approximate μ and how to choose a better b.

We consider $cb(\cdot)$ and by taking c large, we are able to say some quantitative behaviors of such processes.

This is a joint work with

Chii-Ruey Hwang, Brice Franke, Hui-Ming Pei

In the following, we denote

$$T_t^{(c)} f(x) = E_x[f(X^{(c)}(t))],$$

$$L^{(c)} f(x) = \frac{1}{2}\Delta f(x) + cb(x)\nabla f(x).$$

We consider the largest ρ (denoted as $\rho(c)$) such that

$$\int |T_t^{(c)} f(x)|^2 dx \le c_f \exp(-\rho t)$$

for large t and f satisfying

$$\int f(x)dx = 0, \quad \int |f(x)|^2 dx < \infty.$$

Then

$$\rho(c) = \inf\{-Re(\rho); \rho \neq 0 \text{ is in the spectrum of } L^{(c)} \}.$$

$\rho(c)$ is also called the spectral gap

(the gap between 0 and the rest of spectrum)

$\rho(c)$ is used to measure the convergence rate of $X^{(c)}(t)$ to the equilibrium.

$$\int_{\mathbf{T}} |T_t^{(c)} f(x) - \pi(f)|^2 dx \leq c_f \exp(-\rho(c)t).$$

Here

$$\pi(f) = \int_{\mathbf{T}} f(x) dx.$$

The spectral gap for a self-adjoint operator can be expressed by a variational form.

This is the case for $c = 0$.

$$\rho(0) = 2\pi^2 = \inf\{\frac{1}{2}\int_{\mathbf{T}} |\nabla f(x)|^2 dx; \int_{\mathbf{T}} f(x)dx = 0, \\ \int_{\mathbf{T}} f(x)^2 dx = 1\}.$$

We can not have such expression for $\rho(c),\ c > 0$.

This causes great difficulty to calculate $\rho(c)$.

However, we always have

$$\rho(c) \geq \rho(0).$$

Here is a simple argument.

We assume $\pi(f) = 0$. Consider

$$\begin{aligned} \frac{d}{dt}\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx &= 2\int_{\mathbf{T}} T_t^{(c)} f(x) L^{(c)} T_t^{(c)} f(x) dx \\ &= -\int_{\mathbf{T}} \nabla T_t^{(c)} f(x) \nabla T_t^{(c)} f(x) dx \\ &\leq -2\rho(0)\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx. \end{aligned}$$

Then

$$\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx \leq \exp(-2\rho(0)t) \int_{\mathbf{T}} |f(x)|^2 dx.$$

A general discussion is first given in R^d:

Hwang, C.R., Hwang-Ma, S.Y. and Sheu, S.J.(1993), Accelerating Gaussian diffusions, Ann. Appl. Probab.**3** 897-913.

The computation of the spectral gap is done for the following case.

$$U(x) = \frac{1}{2} x \cdot Ax,$$

A is a positive definite matrix.

$$b(x) = SAx,$$

S is a skew symmetric.

We denote ρ_S the spectral gap for $b(x) = SAx$ and $c = 1$.

It is shown that

$$\max\{\rho_S;\ S \text{ is skew symmetric}\} = \frac{1}{d} trac(A).$$

We consider general b here but confine the discussion on torus.

(Our discussion can also be applied to compact manifolds)

Our main result is to show the convergence of $\rho(c)$ when $c \longrightarrow \infty$.

The limit is given by a variational expression as in the following theorem.

Theorem 1.1.

$$\lim_{c\to\infty}\rho(c)=\inf\{\tfrac{1}{2}\int_{\mathbf{T}}|\nabla\psi(x)|^2dx;\exists\mu,b\nabla\psi=i\mu\psi,\\ \int_{\mathbf{T}}\psi(x)dx=0,\int_{\mathbf{T}}|\psi(x)|^2dx=1\}.$$

In this expression, $\psi=\psi_1+i\psi_2$, $i=\sqrt{-1}$.

Using such expression, we give some example to calculate $\rho(c)$ approximately for some b.

Here are some related works.

- Hwang, C.R., Hwang-Ma, S.Y. and Sheu, S.J.(1993), Accelerating Gaussian diffusions, Ann. Appl. Probab.**3** 897-913.
- Hwang, C.R. and Sheu, S.J.(2000), On some quadratic perturbation of Ornstein-Ulenbeck processes, Soochow J. Math. **26** 22-37
- Hwang, C.R, Hwang-Ma, S.Y and Sheu, S.J.(2005), Accelerating diffusions, Ann. Appl. Probab. **15**, 1433-1444.
- Hwang, C.R. and Pei, H. M. (2006), Blowing up Spectral Gap of Laplacian on N-Torus by Antisymmetric Perturbations, preprint.
- Constantin, P., Kislev, A., Ryzhik, L., and Zlatos, A.(2006) Diffusion and mixing in fluid flows, preprint.

- B. Franke, C. R. Hwang, H. M. Pei and S. J. Sheu (2007) The behavior of the spectral gap under growing drift, preprint.

First, we describe our approach. We then give some examples. We then give some details.

We first obtain the result.

(a lower estimate which is an easier part) ,

$$
\begin{aligned}
& \liminf_{c\to\infty} \rho(c) \\
(1.3) \qquad \geq\ & \inf\{\tfrac{1}{2}\int_{\mathbf{T}} |\nabla\psi(x)|^2 dx; \exists \mu, b\nabla\psi = i\mu\psi, \\
& \int_{\mathbf{T}} \psi(x)dx = 0, \int_{\mathbf{T}} |\psi(x)|^2 dx = 1\}.
\end{aligned}
$$

For the upper estimate, we obtain more information about the spectrum of $L^{(c)}$.

To describe, we need some notations.

$$(1.4) \qquad \begin{aligned} H^1 = \ & \{\psi = \psi_1 + i\psi_2; \psi, \nabla\psi \in L^2, \\ & \int_{\mathbf{T}} \psi_1(x)dx = 0 = \int_{\mathbf{T}} \psi_2(x)dx = 0\}, \end{aligned}$$

$$(1.5) \qquad H^1_\mu = \{\psi \in H^1;\ b\nabla\psi = i\mu\psi\},\ \mu \in R.$$

Using these notations, Theorem 1.1 can be stated

(1.6)

$$\begin{aligned} &\lim_{c\to\infty}\rho(c) \\ = \;& \inf\{\tfrac{1}{2}\int|\nabla\psi_1|^2+|\nabla\psi_2|^2dx;\int_{\mathbf{T}}\psi_1^2+\psi_2^2dx=1, \\ &\ \exists\mu \text{ such that } \psi=\psi_1+i\psi_2\in H^1_\mu\}. \end{aligned}$$

Let $\mu\in R$. Assume H^1_μ has nonzero element. Define

$$\rho_\mu=\inf\{\frac{\frac{1}{2}\int_{\mathbf{T}}|\nabla\psi(x)|^2dx}{\int_{\mathbf{T}}|\psi(x)|^2dx};\psi\neq 0\in H^1_\mu\}$$

Theorem 1.2. Let μ,ρ_μ be defined as above. Then for any $r > 0$, there is $c_0 = c_0(r)$ such that for all $c \geq c_0$, there is $-\bar{\rho} + i\bar{\mu}$ in the spectrum of $L^{(c)}$ such that

$$(\bar{\rho} - \rho_\mu)^2 + (\bar{\mu} - c\mu)^2 < r^2.$$

This theorem implies $L^{(c)}$ has an eigenvalue with large imaginary part, and is close to $c\mu$, if $\mu \neq 0$ and H^1_μ contains nonzero elements.

As another consequence of this theorem, we have

$$\limsup_{c \to \infty} \rho(c) \leq \rho_\mu,$$

if H_μ^1 contains nonzero elements. Therefore,

$$\limsup_{c\to\infty} \rho(c) \leq \inf\{\rho_\mu; \mu \in R\}.$$

Together with (1.3), we have Theorem 1.1.

The following are recent results that closely relate to our work.

Relaxation Enhancing

The following result is in the paper Constantin-Kiselev-Ryzhik-Zlatos (2006)

P. Constantin, A. Kiselev, L. Ryzhik and A. Zlatos (2006) Diffusion and Mixing in Fluid Flow, Preprint.,

Theorem 1.3. Assume $b\nabla$ does not have eigenfunction in H^1. That is, there is no $\psi = \psi_1 + i\psi_2 \neq 0 \in H^1$ such that $b\nabla\psi = i\mu\psi$ for some μ. Then for any $t > 0$

$$\|T_t^{(c)}\| \to 0, \ c \to \infty.$$

Here

$$\begin{aligned}\|T_t^{(c)}\|^2 = \ & \sup\{\textstyle\int_{\mathbf{T}} |T_t^{(c)} f(x)|^2 dx; \int_{\mathbf{T}} f(x) dx = 0, \\ & \textstyle\int_{bfT} |f(x)|^2 dx = 1\}\end{aligned}$$

Such result is called relaxation enhancing

The result in Theorem 1.3 implies

$$\lim_{c \to \infty} \rho(c) = \infty.$$

In two dimensional space, if b is smooth, then $b\nabla$ always has eigenfunctions in H^1.

It is difficult to construct b satisfying the condition in Theorem 1.3.

Therefore, it is interesting to study the limit of $\rho(c)$ when the condition in Theorem 1.3 fails.

In the following paper, some particular b is considered and the limit of $\rho(c)$ is calculated:

Hwang, C. R. and Pei, H. M. (2006) Blowing Up the Spectral Gap of Laplacian on N-torus by antisymmetric perturbations(preprint).

Eigenvalue Problem

The following result from H. Berestycki, F. Hamel and N. Nadirashvili (2005) is also relevant

H. Berestycki, F. Hamel and N. Nadirashvili (2005) Elliptic

Eigenvalue Problems with Large Drift and Applications to Nonlinear Propagation Phenomena, Commun. Math. Phys 253, 451-480.

Let b be a divergence free smooth vector field.

Consider the following eigenvalue problem:

$$(1.7) \qquad \begin{array}{l} \frac{1}{2}\Delta\phi + cb\cdot\nabla\phi = \lambda\phi, \ \text{ on } \ D \\ \phi = 0 \ \text{ on } \ \partial D, \end{array}$$

D is a domian in R^d with smooth boundary.

We know there is a unique solution $(\lambda(c), \phi^{(c)})$ such that

$\phi^{(c)}$ is smooth,

$$\phi^{(c)} > 0 \ \text{ on } D, \ \phi^{(c)} = 0 \ \text{ on } \partial D$$

$$\int |\phi^{(c)}(x)|^2 dx = 1.$$

Other solutions have the property that $Re(\lambda) < \lambda(c)$.

$\lambda(c)$ is called the principal eigenvalue.

If $c = 0$, then

$$\lambda(0) = \sup\{\frac{-\int_D \frac{1}{2}|\nabla f(x)|^2 dx}{\int_D |f(x)|^2 dx}\}, \tag{1.8}$$

the supremum is taken over $f \in H_0^1(D)$.

$H_0^1(D)$: those f defined on D with 0 on the boundary and $f, |\nabla f|$ are in $L^2(D)$.

For general c, we have the expression for $\lambda(c)$,

$$\lambda(c) = \sup_{\mu} \inf_{u} \int_D \frac{L^{(c)} u(x)}{u(x)} d\mu(x), \tag{1.9}$$

the supremum is taken over probability measures μ on D, the inf is taken over $u \in C^2(\bar{D}), u > 0$. Here

$$L^{(c)} f = \frac{1}{2} \Delta f + cb \nabla f.$$

Denote

$$H_0 = \{w \in H_0^1(D); b \cdot \nabla w = 0\}$$

The following interesting results in Berestycki-Hamel-Nadirashvili (2005) is close to our (Theorem 1.1)

Theorem 1.4. If $\lambda(c)$ is bounded in c, then H_0 is not empty. And

$$\lim_{c\to\infty} \lambda(c) = \sup\{\frac{-\int_D \frac{1}{2}|\nabla w(x)|^2 dx}{\int_D |w(x)|^2}\}.$$

Here sup is taken over $w \in H_0$.

The probabilistic meaning is the following.

Denote $\tau^{(c)}$ the exit time of $X^{(c)}(t)$ from D,

$$\tau^{(c)} = \inf\{t > 0; X^{(c)}(t) \notin D\}.$$

Then

$$-\lambda(c) = \sup\{k; E_x[\exp(k\tau^{(c)})] < \infty, x \in D\}.$$

Denote

$$S_t^{(c)} f(x) = E_x[f(X^{(c)}(t)), t < \tau^{(c)}].$$

Then the decay rate of $S_t^{(c)} f$ to 0 is given by $\exp(\lambda(c)t)$ as $t \longrightarrow \infty$.

$$\int_0^\infty exp(kt) S_t^{(c)} f(x) dt$$

is finite if and only if $k < -\lambda(c)$.

The following problem may be also interesting.

Let V be smooth function with some growth condition.

We consider

$$E_x[\exp(\int_0^T V(X^{(c)}(t))dt)] \sim \exp(\Lambda(c)T), T \to \infty.$$

Determine the asymptotic behavior of $\Lambda(c)$.

Some study in the following paper may be useful.

H. Kaise and S.J. Sheu (2006), Evaluation of large time expectations for diffusion proceses, preprint

2. Resolvent

The following calculation show some idea for Theorem 1.2.

Let $\lambda > 0$, we consider

$$L^{(c)}\psi^{(c)} - \lambda\psi^{(c)} = -g \tag{2.1}$$

for $g \in L^2$ satisfying $\int g = 0$. This has unique solution,

$$\psi^{(c)}(x) = \int_0^\infty \exp(-\lambda t) T_t^{(c)} g(x) dt.$$

$T^{(c)}g$ is the semigroup generated by $L^{(c)}$.

$$\frac{d}{dt}T_t^{(c)}g = L^{(c)}T_t^{(c)}g.$$

Since

$$\begin{aligned}\tfrac{d}{dt}\int_{\mathbf{T}}|T_t^{(c)}g(x)|^2dx &= \int_{\mathbf{T}}2T_t^{(c)}g(x)L^{(c)}T_t^{(c)}g(x)dx\\ &= -2\int_{\mathbf{T}}|\nabla T^{(c)}g(x)|^2dx\\ &\leq -8\pi^2\int_{\mathbf{T}}|T_t^{(c)}g(x)|^2dx,\end{aligned}$$

we have

$$\int_{\mathbf{T}} |T_t^{(c)} g(x)|^2 dx \le \exp(-8\pi^2 t) \int_{\mathbf{T}} |g(x)|^2 dx.$$

$\psi^{(c}$ given above is well defined even for $\lambda = 0$ or $\lambda = ic\mu$.

In particular, we consider

$$L^{(c)}\psi^{(c)} = ic\mu\psi^{(c)} - g, \tag{2.2}$$

where

$$g = g_1 + ig_2 \in H^1_\mu,$$

$$
(2.3) \qquad \begin{aligned} H^1_\mu = \ & \{g = g_1 + ig_2; \textstyle\int_{\mathbf{T}} |\nabla g|^2 dx < \infty, \\ & \textstyle\int g_1 = \int g_2 = 0,\ b\nabla g = i\mu g\}. \end{aligned}
$$

(2.2) can be rewritten as

$$
\frac{1}{2}\Delta\psi_1^{(c)} + cb\nabla\psi_1^{(c)} = -c\mu\psi_2^{(c)} - g_1,
$$

$$
\frac{1}{2}\Delta\psi_2^{(c)} + cb\nabla\psi_2^{(c)} = c\mu\psi_1^{(c)} - g_2.
$$

Multiplying the first equation by $\psi_1^{(c)}$ and the second equation by $\psi_2^{(c)}$ and adding the relations, we obtain

$$\frac{1}{2}\int_{\mathbf{T}} |\nabla \psi_1^{(c)}|^2 + |\nabla \psi_2^{(c)}|^2 = \int_{\mathbf{T}} \psi_1^{(c)} g_1 + \psi_2^{(c)} g_2. \tag{2.4}$$

Multiplying the first equation by g_1 and the second equation by g_2 and adding the relations, we obtain

$$\frac{1}{2}\int \nabla\psi_1^{(c)}\nabla g_1 + \nabla\psi_2^{(c)}\nabla g_2 - c\int b\nabla\psi_1^{(c)} g_1 - b\nabla\psi_2^{(c)} g_2$$
$$= \int g_1^2 + g_2^2 + c\mu\int \psi_2^{(c)} g_1 - \psi_1^{(c)} g_2.$$

$$b\nabla g_1 = -\mu g_2,\ b\nabla g_2 = \mu g_1.$$

$$(2.5) \qquad \frac{1}{2}\int \nabla\psi_1^{(c)}\nabla g_1 + \nabla\psi_2^{(c)}\nabla g_2 = \int g_1^2 + g_2^2.$$

Here we use

$$\int b\nabla f_1 f_2 = -\int b\nabla f_2 f_1$$

for all f_1, f_2 and g is an element of H_μ^1.

(2.4) implies

$$(\frac{1}{2}\int |\nabla\psi_1^{(c)}|^2 + |\nabla\psi_2^{(c)}|^2)^2 \le (\int g_1^2 + g_2^2)(\int \psi_1^2 + \psi_2^2).$$

(2.5) implies

$$(\int g_1^2 + g_2^2)^2 \le (\frac{1}{2}\int |\nabla \psi_1^{(c)}|^2 + |\nabla \psi_2^{(c)}|^2)(\frac{1}{2}\int |\nabla g_1|^2 + |\nabla g_2|^2).$$

From these two relations, we have

$$(2.6) \qquad \frac{\frac{1}{2}\int |\nabla \psi_1^{(c)}|^2 + |\nabla \psi_2^{(c)}|^2}{\int {\psi_1^{(c)}}^2 + {\psi_2^{(c)}}^2} \le \frac{\frac{1}{2}\int |\nabla g_1|^2 + |\nabla g_2|^2}{\int g_1^2 + g_2^2}.$$

This relation suggests if $g = g_1 + ig_2 \in H_\mu^1$ attains the minimum of

$$(2.7) \qquad \inf\{\frac{\frac{1}{2}\int |\nabla\psi_1|^2 + |\nabla\psi_2|^2}{\int \psi_1^2 + \psi_2^2}\}$$

over $\psi_1 + i\psi_2 \neq 0 \in H^1_\mu$, then limit of $\psi^{(c)}$ (denoted by $\psi^* = \psi_1^* + i\psi_2^*$) is also an element of H^1_μ and attains the minimum of (2.7).

Assume the uniqueness of (2.7) (up to multiplication of constant). Then

$$\psi^* = kg$$

some constant k.

(2.5) implies

$$\frac{1}{k} = \rho_\mu = \text{ the value of (2.7) }.$$

Therefore, we have the picture

$$L\psi^{(c)} = ic\mu\psi^{(c)} - g \sim (-\rho_\mu + ic\mu)\psi^{(c)}.$$

This suggests that an eigenvalue of $L^{(c)}$ close to $-\rho_\mu + ic\mu$ can be found.

Theorem 1.2 gives the precise statement.

The following provides a rigorous argument.

Proposition 2.1. Let $H_\mu \neq \{0\}$ and $g = g_1 + ig_2 \in M_\mu$. Let $\alpha > 0$ be small and $\epsilon, \delta \in R$ be fixed such that

$$0 \neq \epsilon^2 + \delta^2 \leq \alpha^2.$$

Denote $\bar{\rho} = \rho_\mu + \epsilon$ and $\bar{\mu} = c\mu + \delta$. Assume $c_n \to \infty$ be such that there is $\phi_n \in H^1$ satisfying

$$\frac{1}{2}\Delta\phi_n + c_n b \nabla \phi_n = (-\bar{\rho} + i\bar{\mu})\phi_n - g.$$

Then ϕ_n converges to ϕ^* in L^2 and weakly in H^1 as

$n \to \infty$,

$$\phi^* = -\frac{\epsilon + i\delta}{\epsilon^2 + \delta^2} g.$$

Proposition 2.2. Let $H_\mu \neq \{0\}$ and $g = g_1 + ig_2 \in M_\mu$. Let $\alpha > 0$. Assume for each $z = z_1 + iz_2$

$$|z - (-\rho_\mu + ic\mu)|^2 = \alpha^2$$

the following equation has solution $\phi^{z,c}$,

$$\frac{1}{2}\Delta\phi^{z,c} + cb\nabla\phi^{z,c} = z\phi^{z,c} - g.$$

Then for α small, we have

$$\limsup_{c\to\infty} \sup_{|z-(-\rho_\mu+ic\mu)|^2=\alpha^2} \int_{\mathbf{T}} |\phi^{z,c}|^2 < \infty$$

Proof of Theorem 1.2.

Denote $B_\alpha(-\rho_\mu + ic\mu)$ the ball with radius α around $-\rho_\mu + ic\mu$.

$\Gamma_{\alpha,c}$ be the boundary of $B_\alpha(-\rho_\mu + ic\mu)$.

Assume L_c does not have spectrum in $B_\alpha(-\rho_\mu + ic\mu)$.

Then

$$0 = \int_{\Gamma_{\alpha,c}} (L_c - z)^{-1} g dz.$$

$\phi^{z,c} = (L_c - z)^{-1} g$ is the solution of

$$\frac{1}{2}\Delta\phi^{z,c} + cb\nabla\phi^{z,c} = z\phi^{z,c} - g.$$

Then

$$\begin{aligned}
&\textstyle\int_{\mathbf{T}} |2\pi i g(x)| dx \\
&= \textstyle\int_{\mathbf{T}} | \int_{\Gamma_{\alpha,c}} (\phi^{z,c}(x) + \frac{1}{z-(-\rho_\mu + ic\mu)} g(x)) dz | dx \\
&\le \textstyle\int_{\Gamma_{\alpha,c}} \int_{\mathbf{T}} |\phi^{z,c}(x) + \frac{1}{z-(-\rho_\mu + ic\mu)} g(x)| dx dz
\end{aligned}$$

By Proposition 2.1 and 2.2, the righthand side tends to 0. This implies $g = 0$, a contradiction.

In the rest of this section, we give some interesting results about the resolvent of $L^{(c)}$.

Let $\lambda > 0$, we consider

$$L^{(c)}\psi^{(c)} - \lambda\psi^{(c)} = -g$$

for $g \in L^2$ satisfying $\int g = 0$. We denote

$$\psi^{(c)} = R_\lambda^{(c)} g,$$

$$H_0^1 = \{\psi; \int_{bfT} \psi = 0, \int_{\mathbf{T}} |\nabla \psi|^2 < \infty, b\nabla\psi = 0\},$$
$$H^1 = \{\psi; \int_{\mathbf{T}} |\nabla \psi|^2 < \infty\}.$$

Theorem 2.3. For g in L^2, $R_\lambda^{(c)} g$ converges to $R_\lambda^* g$ in H^1. $R_\lambda^* g \in H_0^1$. $R_\lambda^* g$ is the unique element taking the minimum of

$$\int_{\mathbf{T}} (\frac{1}{2}|\nabla\psi(x)|^2 + \lambda\psi(x)^2 - 2g(x)\psi(x))dx,$$

taken over $\psi \in H_0^1$.

Theorem 2.4 R_λ^* defined on H_0 is a family of self-adjoint resolvent operators. That is,

$$R_\lambda^* g - R_\mu^* g = (\mu - \lambda) R_\lambda^* R_\mu^* g, \ \ \lambda > 0, \mu > 0.$$

$$\int_{\mathbf{T}} R_\lambda^* f(x) g(x) dx = \int_{\mathbf{T}} R_\lambda^* g(x) f(x) dx.$$

Theorem 2.5. The range of R_λ^* has closure $\overline{H_0^1}$(in L^2).

From Theorem 2.5, the operator defined by

$$L^* R_\lambda^* g = \lambda R_\lambda^* - g$$

is densely defined on $\overline{H_0^1}$ such that

$$R_\mu^* = (\mu - L^*)^{-1}.$$

L^* generates a semigroup T_t^*.

One expects $T_t^{(c)}$ converges to T_t^*. However, this has not been done rigorously.

Another interesting question is to understand the probabilistic meaning of R_λ^*.

This will connect with the convergence of $X^{(c)}(t)$ defined by

$$dX^{(c)} = cb(X^{(c)})dt + dB(t).$$

3. Examples

Example 1

The following example is considered in Hwang-Pei(2006).

$$b(x) = pcos(2\pi q \cdot x),$$

where

$$p = (p_1, p_2, \cdots, p_d),\ \ q = (q_1, q_2, \cdots, q_d),$$

$p_i, q_i \in Z$ (integer) and

$$0 = p \cdot q = p_1q_1 + p_2q_2 + \cdots + p_dq_d.$$

The limiting value of $\rho(c)$ is then given by

$$\inf\{2\pi^2|m|^2; m = (m_1, m_2, \cdots, m_d),\\ m_i \in Z, p \cdot m = 0\}$$

Take a particular example of $p = (1, M, M^2, \cdots, M^{d-1})$. Then

$$m \cdot p = m_1 + m_2M + m_3M^2 + \cdots + m_dM^{d-1} = 0$$

implies m_1 is a nonzero multiple of M or

$$m_2 + m_3 M + m_4 M^2 + \cdots + m_d M^{d-2} = 0.$$

By this argument, we can see (3.4) has a lower bound $2\pi^2(M^2+1)$. The value

$$m_1 = M, m_2 = -1, m_3 = m_4 = \cdots = m_d = 0$$

gives the lower bound.

We have $H^1_\mu = \{0\}$ if $\mu \neq 0$.

Here is another observation.

In this example, let

$$m \cdot p = 0, m = (m_1, m_2, \cdots, m_d), m_i \in Z.$$

Then $L^{(c)}$ has eigenfunction $\phi_m(x) = \exp(2\pi m \cdot x)$,

$$L^{(c)}\phi_m = -2\pi^2 |m|^2 \phi_m.$$

Example 2

We consider a modification of previous example.

Let $d = 3$. General d is similar.

$$b(x) = (2\pi M sin(2\pi x_3), 2\pi M cos(2\pi x_3), 1),$$

M is a positive integer.

The limit of $\rho(c)$ is equal to $2\pi^2(1 + 2\pi^2 M^2)$.

$H^1_\mu \neq \{0\}$ iff $\mu = 2\pi k$, k is an integer.

$\rho_\mu = 2\pi^2(1 + 2\pi^2 M^2) + 2\pi^2 k^2$ if $\mu = 2\pi k$.

It is difficult to construct eigenfunctions of $L^{(c)}$.

More examples on manifold from geometry can be found in our paper.

4. Concluding Remark

Here are several problems for further consideration.

- How to obtain our main results by using probability methods such as coupling?

- Obtain K_c such as

$$\|T_t^{(c)} f - \pi(f)\| \leq K_c \|f - \pi(f)\| \exp(-\rho(c)t).$$

  This will give another proof for the result in Constantin-

Kislev-Ryzhik-Zlatos.

- More generally, take $c = 1$. Study quantitatively how the spectral gap depend on b.
- Obtain similar results in R^d.
- The limiting process of $X^{(c)}(t)$ should be a process with state space $\mathcal{E}$, the collection of integral curves of
$$\frac{d}{dt}X(t) = b(X(t)).$$
How to describe the process?