Functional central limit theorem for spatial birth and death processes

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Outline

1. Introduction

2. An Extended Penrose's Multivariate Spatial CLT
   - Theorem
   - Sketch of the proof

3. Functional CLT for spatial birth-death process
   - Theorem
   - Proof
   - Application to random packing problem

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Spatial birth-death process

- Identify a countable subset of $\mathbb{R}^d$ with a counting measure η given by assigning unit mass to each point.
- $\mathcal{N}(\mathbb{R}^d)$: the collection of all counting measures on $\mathbb{R}^d$.
- Spatial birth-death process is a continuous time Markov process. The state space is some subset of $\mathcal{N}(\mathbb{R}^d)$.
- Generator of the process is of the form:

$$\mathcal{A}f(\eta) = \int_{\mathbb{R}^d} (f(\eta + \delta_x) - f(\eta))\lambda(u,\eta)du + \int_{\mathbb{R}^d} (f(\eta - \delta_x) - f(\eta))\delta(x,\eta)\eta(dx),$$

for f in an appropriate domain.

Spatial birth-death process

- Birth rate: $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to [0, +\infty)$. If the point configuration at time t is $\eta \in \mathcal{N}(\mathbb{R}^d)$, then the probability that a point in a set $A \subset \mathbb{R}^d$ is added to the configuration in the next time interval of length Δt is approximately $\int_A \lambda(x, \eta) dx \Delta t$.
- Death rate: $\delta : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to [0, +\infty)$. The probability that a point $x \in \eta$ is deleted from the configuration in the next time interval of length Δt is approximately $\delta(x, \eta)\Delta t$. In this paper, we assume δ is a nonnegative constatnt for our spatial birth-death processes. If $\delta = 0$, it is spatial pure birth process.

Spatial birth-death process

Example: Random packing problem. Consider the following prototype random packing model. Suppose that unit volume open balls arrive randomly. The centers of the incoming balls have a homogeneous space-time Poisson distribution in $\mathbb{R}^d \times [0, \infty)$. An incoming ball arriving at time t is packed if it does not overlap any balls already packed before time t, otherwise it is discarded. Let $\eta_t(B)$ denote the number of balls packed before and at time t and in B, where B is any measurable set in $\mathbb{R}^d$. Then η_t is a spatial pure birth process with birth rate

$$\lambda(x, \eta) = \mathbf{1}_{\{\min_{y \in \eta} |x - y| \geq 2r_d\}},$$

where r_d is the radius of the unit volume ball in $\mathbb{R}^d$.

Purpose and main ideas

- **Starting point:** Prove the CLT for $\eta_t(lA)$ as $l \to \infty$, where A is a bounded measurable subset of $\mathbb{R}^d$. In the random packing problem, $\eta_t(lA)$ is the total number of packed balls before and at time t in the region lA.
- We also can use the similar idea to prove the CLT for $\eta(lA)$, where η is a spatial point process the distribution of which is the stationary distribution of a spatial birth-death process. This result can be used to prove the asymptotic normality for time-invariance estimations.

Purpose and main ideas

- Because $\eta_t(lA)$ can be written as $\int_{\mathbb{R}^d} \mathbf{1}_A(\frac{x}{l})\eta_t(dx)$, where $\mathbf{1}_A(x)$ is an indicator function of A, an interesting problem is whether we can prove the CLT for $\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)$, where f is a more general function.
- If it is true for some f, a further question is to decide whether the real-valued process $\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)$ converges weakly to some process. The answer is positive.

Purpose and main ideas

For each $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, define

$$Y^{(l)}(f,t) = \frac{1}{l^{d/2}}\Big[\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)\Big], l \geq 1.$$

We will give some conditions on birth rate such that there exists a Gaussian process W indexed by $(L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \times [0,\infty)$. $W(f,\cdot)$ has sample paths in $C_{\mathbb{R}}[0,\infty)$. Let $\{f_1,\cdots,f_n\}$ be a finite collection of the functions in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the family of stochastic processes

$$\{(Y^{(l)}(f_1,t),\cdots,Y^{(l)}(f_n,t)) : l \geq 1\},$$

converges weakly to $(W(f_1,\cdot),\cdots,W(f_n,\cdot))$ as $l \to \infty$ in $D_{\mathbb{R}^n}[0,\infty)$. We get the same theoem for spatial pure birth processes.

Purpose and main ideas

Main ideas.

- Prove the multivariate CLT for indicator functions of bounded measurable subsets with boundary having Lebesgue measure zero (Riemann measurable subsets).
- In order to do that, we need to extend Penrose's Theorem (2005), then apply it to the SDE in Kurtz and Garcia (2006), where the spatial birth and death processes were obtained as the solution of the SDE.

Purpose and main ideas

Main ideas (Cont.)

- Use the linear combinations of indictor functions of Riemann measurable sets to approximate a general bounded and integrable function f. Then we can prove the weak convergence of finite-dimensional distributions of $Y^{(l)}(f,t)$.
- Prove the relative compactness of

$$\{(Y^{(l)}(f_1,t),\cdots,Y^{(l)}(f_n,t)) : l \geq 1\}.$$

Spatial birth-death process as the solution of SDE

- In Garcia and Kurtz(2006), Spatial birth-death processes are obtained as solutions of a system of stochastic equations.
- The state space and its topology:
  - $K_1 \subset K_2 \subset \cdots$ satisfy $\cup_k K_k = \mathbb{R}^d$.
  - $\{c_k : k = 1, \cdots\}$ bounded continuous functions on $\mathbb{R}^d$. $c_k \geq 0$ and $\inf_{x \in K_k} c_k(x) > 0$.
  - $\mathcal{S} = \{\xi \in \mathcal{N}(\mathbb{R}^d) : \int_{\mathbb{R}^d} c_k(x)\xi(dx) < \infty, k = 1, 2, \cdots\}$.
  - $\mathcal{C}$=$\{f \in \bar{C}(\mathbb{R}^d) : |f| \leq ac_k$ for some k and $a > 0\}$. Topologize $\mathcal{S}$ by the weak* topology generated by $\mathcal{C}$.
  - $D_{\mathcal{S}}[0, \infty)$ is the space of cadlag $\mathcal{S}$-valued functions with the Skorohod (J_1) topology.

Spatial birth-death process as the solution of SDE

- The spatial birth-death process satisfies

$$\eta_t(A) = \int_{A\times[0,t]\times[0,+\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx,ds,dr,du) + \int_{A\times[0,\infty)} \mathbf{1}_{(t,\infty)}(r)\hat{\eta}_0(dx,dr). \quad (1.1)$$

- N: a Poisson random measure on $\mathbb{R}^d \times [0,\infty)^3$ with mean measure $dx \times ds \times e^{-r}dr \times du$.
- $\hat{\eta}_0$: the point process on $\mathbb{R}^d \times [0,\infty)$ independent of N obtained by associating to each “count” in η_0 an independent, unit exponential random variable.

Spatial birth-death process as the solution of SDE

The following two conditions given in Garcia and Kurtz (2006) guarantee the integral on the right of (1.1) exists and determines an $\mathcal{S}$-valued process with sample paths in $D_{\mathcal{S}}[0,\infty)$.

- **Condition 1**: For each compact $\mathcal{K} \subset \mathcal{S}$, $sup_{\zeta\in\mathcal{K}} \int_{\mathbb{R}^d} c_k(x)\lambda(x,\zeta)dx < \infty$, $k = 1, 2, \cdots$.
- **Condition 2**: If $\lim_{n\to\infty} \int_{\mathbb{R}^d} c_k(x)|\zeta_n - \zeta|(dx) = 0$ for each $k = 1, 2, \cdots$, then $\lambda(x,\zeta) = \lim_{n\to\infty} \lambda(x,\zeta_n)$.

Spatial birth-death process as the solution of SDE

Theorem (Garcia and Kurtz (2006))

Assume that λ is translation invariant and satisfies Condition 1 and 2, and that η_0 is translation invariant. Suppose that

$$a(x,y) \geq \sup_{\eta} |\lambda(x,\eta+\delta_y) - \lambda(x,\eta)| \tag{1.2}$$

and that there exists a positive and bounded function $c'(x)$ such that

$$M = \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c'(x)a(x,y)}{c'(y)} dy < \infty, \tag{1.3}$$

Then, there exists a unique solution of (1.1) *and η_t is translation invariant.*

Remarks.

(a) In the random packing problem, we can use $a(x,y) = \mathbf{1}_{\{|x-y|<2r_d\}}$.

(b) In our main theorem, the birth rate is assumed bounded and

$$a(x,y) \le \frac{b}{1 + |x-y|^{2d+\delta}}.$$

In this case, we can set $c'(x) = 1$, then all the conditions on c' in above theorem are satisfied.

(c) In this paper, we always assume that $\lambda(x, \zeta)$ is bounded. Hence if $c_k(x)$ is integrable, Condition 1 is satisfied.

(d) By Lemma 2.5 in Kurtz and Garcia (2006), if there is a finite interaction range, that is, there exists a large number K such that $a(x, y) = 0$ for all x, y satisfying $|x - y| > K$, Condition 2 is satisfied. The random packing problem satisfies this condition.

(e) If

$$a(x,y) \leq \frac{b}{1+|x-y|^{2d+\delta}}.$$

we can set one of the functions c_k in Condition 1 and 2, say c_1, to be $\frac{b}{1+|y|^{2d+\delta}}$, then for each x, we can find a number $k(x)$ which only depends on x, such that

$$a(x,y) \leq k(x)c_1(y),$$

for all y. Then in this case, Condition 2 is satisfied. In fact, if $\lim_{n\to\infty}\int_S c_1(x)|\zeta_n-\zeta|(dx) = 0$,

$$\begin{aligned} &|\lambda(x,\zeta)-\lambda(x,\zeta_n)| \\ \leq &\int_S a(x,y)|\zeta_n-\zeta|(dy) \leq k(x)\int_S c_1(y)|\zeta_n-\zeta|(dy) \to 0 \end{aligned}$$

One can see that Condition 1 and 2 are not restrictive in this paper.

An Extended Penrose's Multivariate Spatial CLT

Notations:

- d: dimension of Euclidean space.
- For $x = (x_1, \cdots . x_d) \in \mathbb{R}^d$, write $|x|$ for the Euclidean norm of x and $\|x\| := \max_{1 \le i \le d} |x_i|$.
- For $A \subseteq \mathbb{R}^d$, $t \in R$, and $y \in \mathbb{R}^d$,
  - tA=$\{tx \colon x \in A\}$; $\tau_y(A)$=$\{y + x \colon x \in A\}$; $\partial(A)$=boundary of A.
  - If A is (Lebesgue) measurable, write $|A|$ for its Lebesgue measure.
- **Riemann measurable set**: bounded and with Lebesgue-null boundary.
- $\mathcal{R}(\mathbb{R}^d)$ denotes the collection of Riemann measurable subsets of $\mathbb{R}^d$.

An Extended Penrose's Multivariate Spatial CLT

- $X = (X_z, z \in \mathbb{Z}^d)$: a family of independent identically distributed random elements. For example, in this paper

$$X_z = (\tau_{-z}(N|_{(B_0+z)\times[0,\infty)^3}), \tau_{-z}(\hat{\eta}_0|_{(B_0+z)\times[0,\infty)})),$$

  τ_{-z} denotes the shift operator, B_0 is the unit cube in $\mathbb{R}^d$.
- $H = \{H_l(X, A) : l \geq 1, A \in \mathcal{R}(\mathbb{R}^d)\}$: a collection of random variables, where for each $l \geq 1$ and A, $H_l(X, A)$ is a function of $(X_z, z \in \mathbb{Z}^d)$. We call H random set function. Sometimes we write $H_l(X, A)$ as $H_l(A)$.
- For example, in this paper, $H_l(A) = \eta_t(lA)$. Another example is

$$H_l(A) = \int_{lA}\int_{lA} f(x, y)\eta_t(dx)\eta_t(dy),$$

  where f some appropriate function.

An Extended Penrose's Multivariate Spatial CLT

- For $y, z \in \mathbb{Z}^d$, write $y \prec z$ if y precedes z in the lexicographic ordering on $\mathbb{Z}^d$, and $y \preccurlyeq z$ if either $y \prec z$ or $y = z$. $\mathcal{F}_y = \bigvee_{z \in \mathbb{Z}^d, z \preccurlyeq y} \sigma\{X_z\}$.
- For $l \geq 1$, $y \in \mathbb{Z}^d$, define $H_{l,y}(A) := H_l(\tau_y X, A)$, where $\tau_y X$ denotes the family $(X_{z+y}, z \in \mathbb{Z}^d)$. The definition of $H_{l,y}(A)$ in terms of $\tau_y X$ is the same as the definition of $H_l(A)$ in terms of X
- X_*: a copy of $X_{\mathbf{0}}$ and independent of the family X.
- For any $y \in \mathbb{Z}^d$, define X^y to be the family X with the value X_y at y replaced by X_*, but with the values at all other sites the same.
- $\Delta^H_{l,y}(A) = H_{l,y}(X, A) - H_{l,y}(X^{\mathbf{0}}, A)$.

Stabilization Conditions: There exists a random variable Δ^H_∞ such that for any sequence $\{(l_n, y_n) | l_n \geq 1, y_n \in \mathbb{Z}^d\}$ and any $A \in \mathcal{R}(\mathbb{R}^d)$,

$$\Delta^H_{l_n, y_n}(A) \xrightarrow{P} \Delta^H_\infty \quad \text{if} \quad \liminf_{n\to\infty}(\tau_{y_n}(l_n A)) = \mathbb{R}^d \tag{2.1}$$

$$\Delta^H_{l_n, y_n}(A) \xrightarrow{P} 0 \quad \text{if} \quad \liminf_{n\to\infty}(\tau_{y_n}(l_n (A^c))) = \mathbb{R}^d \tag{2.2}$$

and for each $A \in \mathcal{R}(\mathbb{R}^d)$, there exists $K > 0$ (depending on A), such that

$$\lim_{l\to\infty} \frac{1}{l^d} \sum_{y\in\mathbb{Z}^d, \|y\|\geq lK} E\left[(\Delta^H_{l,y}(A))^2\right] = 0. \tag{2.3}$$

Moment Condition: There exists $\gamma > 2$ such that

$$\sup\{E\left[|\Delta^H_{l,-y}(A)|^\gamma\right] : A \in \mathcal{R}(\mathbb{R}^d)), l \geq 1, y \in \mathbb{Z}^d\} < \infty. \tag{2.4}$$

Theorem(Extended Penrose's Multivariate Spatial CLT).
Suppose that $H^1, \cdots, H^k$ are random set functions and are integrable for each $l \geq 1$ and $A \in \mathcal{R}(\mathbb{R}^d)$. Each of them satisfies the stabilization conditions and the moment condition for some $\gamma > 2$. Let the $k \times k$ matrix $(\sigma_{ij}^*)_{i,j=1}^k$ be given by

$$\sigma_{i,j}^* := E\left[E(\Delta_\infty^{H^i}|\mathcal{F}_\mathbf{0})E(\Delta_\infty^{H^j}|\mathcal{F}_\mathbf{0})\right],$$

where $\mathbf{0}$ is the origin in $\mathbb{R}^d$. Then if $A_1, \cdots, A_k \in \mathcal{R}(\mathbb{R}^d)$, we have

$$\lim_{l\to\infty} l^{-d} Cov(H_l^i(A_i), H_l^j(A_j)) = \sigma_{i,j}^* |A_i \cap A_j|$$

and as $l \to \infty$,

$$(l^{-d/2}(H_l^i(A_i) - EH_l^i(A_i)))_{i=1}^k \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \sigma_{i,j}^* |A_i \cap A_j|)_{i,j=1}^k.$$

An Extended Penrose's Multivariate Spatial CLT

Remark. In Penrose (2005), the stabilization conditions only contained (2.1) and (2.2), however, $H_l(A)$ is required to be a function of $(X_z : z \in lC_0)$, where C_0 is a Riemann measurable subset of $\mathbb{R}^d$, that is, $H_l(A)$ only depends on a finite collection of elements in $(X_z : z \in \mathbb{Z}^d)$. In order to remove this restriction, we add one more stabilization condition (2.3).

Sketch of the proof

- Focus on the proof of the theorem for one-dimensional case.
- For multidimensional case, we can prove the CLT for any linear combination by the same martingale argument as the one-dimensional case. Then by the Cramer-Wold device, we can get the multidimensional CLT.

$$
\begin{aligned}
&l^{-d/2}(H_l(A) - EH_l(A)) \\
=&l^{-d/2}(H_l(A) - E[H_l(A)|\mathcal{H}_{[lK]}]) + l^{-d/2}(E[H_l(A)|\mathcal{H}_{[lK]}] - EH_l(A)),
\end{aligned}
$$

where $[lK]$ denotes the largest integer number less than or equal to lK. By calculation,

$$
\begin{aligned}
&l^{-d}E(E[H_l(A)|\mathcal{H}_{[lK]}] - E[H_l(A)])^2 \\
\leq&\frac{1}{l^d}\sum_{y\in\mathbb{Z}^d, \|y\|\geq lK} E\left[(\Delta^H_{l,y}(A))^2\right],
\end{aligned}
$$

which goes to zero by (2.3).
We will show $l^{-d/2}(H_l(A) - E[H_l(A)|\mathcal{H}_{[lK]}])$ converges weakly to the required normal distribution as $l \to \infty$.

Theorem (McLeish,1974)

Let $\{X_{n,i} : 1 \leq i \leq k_n, n = 1, 2 \cdots\}$ *be a martingale difference array satisfying*
(a)$max_{i \leq k_n} |X_{n,i}|$ *is uniformly bounded in* L^2.
(b)$max_{i \leq k_n} |X_{n,i}| \rightarrow 0$ *in probability.*
(c)$\sum_{i=1}^{k_n} X_{n,i}^2 \rightarrow 1$ *in probability.*
Then $S_n = \sum_{i=1}^{K_n} X_{n,i} \rightarrow N(0, 1)$ *weakly.*

- Let $\{z_1, \cdots, z_{n_l}\}$ be the set of all the points $z \in \mathbb{Z}^d$ such that $\|z\| < [lK]$ ordered by lexicographic ordering on $\mathbb{Z}^d$ from the smallest to the largest.
-

$$
\begin{aligned}
& H_l(A) - E[H_l(A)|\mathcal{H}_{[lK]}] \\
= & \sum_{i=1}^{n_l} (E[H_l(A)|\mathcal{H}_{[lK]} \bigvee_{1 \le j \le i} \sigma\{X_{z_j}\}] - E[H_l(A)|\mathcal{H}_{[lK]} \bigvee_{1 \le j \le i-1} \sigma\{X_{z_j}\}]) \\
= & \sum_{i=1}^{n_l} D^l_{z_i},
\end{aligned}
$$

where $\{D^l_{z_i} | i = 1, \cdots, n_l\}$ are martingale differences.

Let $f(x)$ be a bounded and integrable measurable function on $\mathbb{R}^d$ (with respect to Lebesgue measure), define a family of real processes

$$Y^{(l)}(f,t) = \frac{1}{l^{d/2}}\Big[\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)\Big], l \geq 1.$$

Functional CLT for spatial birth-death process

Theorem

Assume that the birth rate λ is bounded by some positive number L, translation invariant, and satisfies Conditions 1 and 2. Suppose that η_0 is a Poisson random measure on $\mathbb{R}^d$ with constant intensity μ_1 and is independent of N and that there exists a positive function $c(x)$ such that

$$M = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(x)a(x,y)}{c(y)} dy < \infty, \qquad (3.1)$$

$c(x)$ is bounded in a neighborhood of the origin in $\mathbb{R}^d$, and $\int_{\mathbb{R}^d} \frac{1}{c(x)^{\frac{1}{3}}} dx < \infty$.

Functional CLT for spatial birth-death process

Theorem (Cont.)

Assume that

$$a(x,y) \leq \frac{b}{1+|x-y|^{2d+\delta}}, \tag{3.2}$$

for some constants $b > 0$ *and* $\delta > 0$. *Then there exists a Gaussian process* W *indexed by* $(L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \times [0,\infty)$, *such that for each* $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $W(f,\cdot)$ *has sample paths in* $C_{\mathbb{R}}[0,\infty)$. *Let* $\{f_1,\cdots,f_n\}$ *be a finite collection of the functions in* $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, *the family of stochastic processes*

$$\{(Y^{(l)}(f_1,t),\cdots,Y^{(l)}(f_n,t)) : l \geq 1\},$$

converges weakly to $(W(f_1,\cdot),\cdots,W(f_n,\cdot))$ *as* $l \to \infty$ *in* $D_{\mathbb{R}^n}[0,\infty)$.

Remarks. (a). If the inequality (3.2) is satisfied for some constant $\delta > d$, all the conditions on $c(x)$ can be derived from (3.2). In fact, we can take $c(x) = 1 + |x|^{2d+\delta}$. Then $\frac{1}{c(x)^{\frac{1}{3}}}$ is integrable, and

$$\begin{aligned} M &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(x)a(x,y)}{c(y)} dy \\ &= \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(1 + |x|^{2d+\delta})a(x,y)}{(1 + |y|^{2d+\delta})} dy \\ &\leq 2^{2d+\delta} b \int_{\mathbb{R}^d} \frac{1}{(1 + |y|^{2d+\delta})} dy < \infty. \end{aligned}$$

Remarks.

(b). If $c(x)$ is a positive function satisfying the conditions in Theorem 2.2 and $a(x,y)$ satisfies (3.2), then $c(x) \vee 1 = \max\{c(x), 1\}$ also satisfies the conditions in Theorem 2.2. In fact, defining $\tilde{c}(x) = c(x) \vee 1$, we have

$$\int_{\mathbb{R}^d} \frac{1}{\tilde{c}(x)^{\frac{1}{3}}} dx \leq \int_{\mathbb{R}^d} \frac{1}{c(x)^{\frac{1}{3}}} dx < \infty,$$

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\tilde{c}(x) a(x,y)}{\tilde{c}(y)} dy < \infty.$$

Hence, we can choose $c(x)$ to be a function bounded from below by a positive number, so $\frac{1}{c(x)}$ is bounded, and hence $\frac{1}{c(x)^\alpha}$ is integrable for all $\alpha \geq \frac{1}{3}$.

Functional CLT for spatial birth-death process

- E: a subset of the collection of all bounded and integrable Lebesgue measurable functions such that E can be topologized to be a nuclear Frechet space, and for any $f \in E$, there exists $c_k(x)$ and a positive number a such that $|f| \leq ac_k$.
- For example, if each c_k has polynomial rate-of-convergence to zero as $|x| \to \infty$, then E can be the Schwartz space in $\mathbb{R}^d$. Let E' be its dual space.

Corollary

Suppose all the conditions in Theorem 2.2 are satisfied. Let

$$\xi_t^{(l)}(A) = \frac{1}{l^{d/2}}\Big[\eta_t(lA) - E\eta_t(lA)\Big], \quad l \geq 1,$$

be a family of E'-valued processes whose paths are the elements in $D_{E'}[0,\infty)$, then there exists a E'-valued process ξ whose paths are the elements in $C_{E'}[0,\infty)$ such that $\xi^{(l)} \to \xi$ weakly.

Proof.

The result follows immediately from our Functional CLT and Theorem 5.3 in Mitoma (1983). □

Proof of Functional CLT

- We try to use the Extended Penrose's CLT to prove that the finite-dimensional distributions of $Y^{(l)}(1_A, \cdot)$ converge weakly to multivariate normal distributions, where A is a Riemann measurable subset.
- For any set $E \subset \mathbb{R}^d \times [0, \infty)^3$(or $\mathbb{R}^d \times [0, \infty)$), we use $N|_E$(or $\hat{\eta}_0|_E$) to denote the restriction of N(or $\hat{\eta}$) to E.
- Let $B_0 = \prod_{i=1}^d [-\frac{1}{2}, \frac{1}{2})$ be the unit cube with the center at the origin in $\mathbb{R}^d$.
- We define a family of independent identically distributed random elements $X = (X_z : z \in \mathbb{Z}^d)$, where

$$X_z = (\tau_{-z}(N|_{(B_0+z)\times[0,\infty)^3}), \tau_{-z}(\hat{\eta}_0|_{(B_0+z)\times[0,\infty)})),$$

τ_{-z} denotes the shift operator.

Proof of Functional CLT

- N_0: a Poisson random measure on $B_0 \times [0, \infty)^3$ independent of N that has the same distribution as $N|_{B_0 \times [0,\infty)^3}$.
- Let N^0 be the Poisson random measure obtained from N by replacing $N_{B_0 \times [0,\infty)^3}$ with N_0.
- $\hat{\eta}_{0,0}$: a random measure independent of $\hat{\eta}_0$, having the same distribution as $\hat{\eta}_0|_{B_0 \times [0,\infty)}$.
- Let $\hat{\eta}_0^0$ be $\hat{\eta}_0$ with the restriction to $B_0 \times [0, \infty)$ replaced by $\hat{\eta}_{0,0}$.
- Let $X_* = (N_0, \hat{\eta}_{0,0})$, then X_* has the same distribution as X_0 and is independent of (X_z).
- Let $X^{\mathbf{0}}$ be the family X with X_0 replaced by X_* and with all others the same.

Proof of Functional CLT

- For any $0 \le t_1 \le \cdots \le t_n$ and Riemann measurable sets $A_1, \cdots, A_n$, we will show that $(Y^{(l)}(\mathbf{1}_{A_1}, t_1), \cdots, Y^{(l)}(\mathbf{1}_{A_n}, t_n))$ converges weakly to a multivariate normal distribution.
- Under the conditions of our Functional CLT, η_t is the unique solution of the following SDE,

$$\eta_t(A) = \int_{A\times[0,t]\times[0,+\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx, ds, dr, du) + \int_{A\times[0,\infty)} \mathbf{1}_{(t,\infty)}(r)\hat{\eta}_0(dx, dr),$$

then η_t is the function of N and $\hat{\eta}_t$, and hence it is a function of the family X.

Proof of Functional CLT

- Let $H_l(X, A) = \eta_t(lA)$, where $t \geq 0$ and A is a Riemann measurable set in $\mathbb{R}^d$.
- For any $z \in \mathbb{Z}^d$, $H_{l,z}(X, A) = H_l(\tau_z(X), A) = \eta_t(lA + z)$. Then,

$$\Delta_{l,z}^H(A) = H_{l,z}(X, A) - H_{l,z}(X^{\mathbf{0}}, A) = \eta_t(lA + z) - \eta_t^0(lA + z),$$

where η_t^0 is the unique solution of the above SDE with N replaced by N^0.

Key Lemma

Lemma

For any $t \geq 0$, we have

$$\sup_{x \in \mathbb{R}^d} c(x) E\left[\int_{\mathbb{R}^d} a(x,y)|\eta_t - \eta_t^0|(dy)\right] \leq 2(L + \mu_1) M \sup_{y \in B_0} c(y) e^{tM}.$$

Remark: The function $c(x)$ in the conditions of Theorem 2.2 is bounded in a neighborhood of the origin in $\mathbb{R}^d$, without loss of generality, here we assume that $c(x)$ is bounded in B_0, because otherwise we can replace the lattice $\mathbb{Z}^d$ by $\epsilon\mathbb{Z}^d$ for a sufficiently small $\epsilon > 0$.

Proof of Functional CLT

The moment condition (2.4) follows from the next lemma.

Lemma

Suppose $\int_{\mathbb{R}^d} \frac{1}{c(x)^{\frac{1}{3}}} dx < \infty$*. We have*

$$\sup\left\{E\left[(|\eta_t - \eta_t^0|(B))^3\right] : B \quad \textit{is a bounded measurable set in } \mathbb{R}^d\right\} < \infty,$$

Proof of Functional CLT

Verification of the first stabilization condition (2.1):

- By calculation, we can get the following inequality. For any bounded measurable set B,

$$E\left[|\eta_t - \eta_t^0|(B)\right] \leq K_t^{(1)}\left(\int_B \frac{1}{c(x)}dx + \int_{B\cap B_0} dx\right), \tag{3.3}$$

where $K_t^{(1)}$ is a constant which depends on t, but does not depend on B.

- Let $\{C_n\}$ be a sequence of Riemann measurable subsets of $\mathbb{R}^d$ satisfying $\liminf_{n\to\infty} C_n = \mathbb{R}^d$. Then

$$\begin{aligned}&\left|E\left[(\eta_t(C_n)-\eta_t^0(C_n))-(\eta_t(C_m)-\eta_t^0(C_m))\right]\right|\\ \le& E\left[|\eta_t-\eta_t^0|(C_n\setminus C_m)\right]+E\left[|\eta_t-\eta_t^0|(C_m\setminus C_n)\right]\\ \le& K_t^{(1)}\left(\int_{C_n\Delta C_m}\frac{1}{c(x)}dx+\int_{(C_n\Delta C_m)\cap B_0}dx\right)\to 0,\quad \text{as}\quad m,n\to\infty.\end{aligned}$$

- Hence $\Delta^H_{l_n,z_n}(A)=\eta_t(l_nA+z_n)-\eta_t^0(l_nA+z_n)$ converges in L^1 if $\liminf_{n\to\infty}(\tau_{y_n}(l_nA))=\mathbb{R}^d$, and the limit does not depend on A and the sequence (l_n,z_n).
- The second stabilization condition (2.2) can be similarly verified.

Now we show that the third stabilization condition is true.

- $$E\left[(|\eta_t - \eta_t^0|(B))^2\right] \leq K_t^{(2)}[(\int_B \frac{1}{c(x)^{\frac{1}{2}}}dx)^2 + \int_B \frac{1}{c(x)^{\frac{1}{2}}}dx + (\int_{B\cap B_0} dx)^2 + \int_{B\cap B_0} dx],$$

  where $K_t^{(2)}$ is a constant which depends on t, but does not depend on B.
- We can pick K large enough such that

  $$\frac{1}{l^d}\sum_{y\in\mathbb{Z}^d, \|y\|\geq lK} E\left[(\Delta_{l,y}^H(A))^2\right] \leq K_t^{(2)}(\frac{K}{2})^d \int_{\|x\|\geq\frac{lK}{2}} \frac{1}{c(x)^{\frac{1}{2}}}dx \to 0 \quad \text{as} \quad l \to \infty,$$

  and third stabilization condition follows.

Theorem

Assume that the birth rate λ is bounded by some positive number L and translation invariant, and satisfies Conditions 1 and 2. Suppose that η_0 is a Poisson random measure on $\mathbb{R}^d$ with constant intensity independent of N and that there exists a positive function $c(x)$ such that (1.3) *is satisfied, and $c(x)$ is bounded in a neighborhood of the origin in $\mathbb{R}^d$, and $\int_{\mathbb{R}^d} \frac{1}{c(x)^{\frac{1}{3}}} dx < \infty$. For any $0 \leq t_1 \leq \cdots \leq t_n$ and Riemann measurable sets $A_1, \cdots, A_n$, $(Y^{(l)}(\mathbf{1}_{A_1}, t_1), \cdots, Y^{(l)}(\mathbf{1}_{A_n}, t_n))$ converges weakly to a multivariate normal distribution and the covariance of the limit distribution can be given in terms of Δ_∞ as in Theorem 2.1.*

- If $f(x)$ is the linear combinations of indicator functions of Riemann measurable sets in $\mathbb{R}^d$, then the finite-dimensional distributions of $Y^{(l)}(f,\cdot)$ converge weakly to multivariate normal distributions.
- If f is bounded integrable function, we can find a sequence $\{f_n(x) : n = 1, \cdots\}$ such that f_n is a linear combinations of indicator functions of Riemann measurable sets, and

$$\sup_n \|f_n\| < \infty \quad \text{and} \quad \lim_n \int_{\mathbb{R}^d} |f_n(x) - f(x)|dx = 0, \tag{3.4}$$

then we have

$$\lim_n \int_{\mathbb{R}^d} (f_n(x) - f(x))^2 dx = 0.$$

Proof of Functional CLT

We can show that for any $t \geq 0$,

$$\sup_l E\Big[\frac{1}{l^{d/2}}\Big(\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)\Big) - \frac{1}{l^{d/2}}\Big(\int_{\mathbb{R}^d} f_n(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f_n(\frac{x}{l})\eta_t(dx)\Big)\Big]^2 \to 0,$$

as $n \to \infty$, so the finite-dimensional distributions of $Y^{(l)}(f, \cdot)$ converge weakly to multivariate normal distributions.

Proof of Functional CLT

Lemma

$$e^t \int_{\mathbb{R}^d \times [0,\infty)} f(\frac{x}{l}) \mathbf{1}_{(t,\infty)}(r) \hat{\eta}_0(dx, dr), t \geq 0,$$

is an $(\mathcal{F}_t)$*-martingale for any* $l \geq 1$*,* $(\mathcal{F}_t)$ *is the filtration generated by* N *and* $\hat{\eta}_0$*. And its quadratic variation is*

$$\int_{\mathbb{R}^d \times [0,\infty)} e^{2r} f^2(\frac{x}{l}) \mathbf{1}_{[0,t]}(r) \hat{\eta}_0(dx, dr), t \geq 0.$$

Proof of Functional CLT

Lemma

Suppose λ is bounded by a constant L and $a(x,y) \le \frac{b}{1+|x-y|^{2d+\delta}}$, for some constants $b > 0$ and $\delta > 0$. Then

$$\sup_{x,y\in\mathbb{R}^d,s,t\le T} (1 \vee |x-y|^{d+\delta}) \times \left| E\left[(\lambda(x,\eta_s) - E\lambda(x,\eta_s))(\lambda(y,\eta_t) - E\lambda(y,\eta_t)) \right] \right| < \infty,$$

for any $T > 0$, where $1 \vee |x-y|^{d+\delta}$ denotes $\max\{1, |x-y|^{d+\delta}\}$.

We use a kind of “coupling” method to prove this lemma.

Proof of Functional CLT

Lemma

Suppose $a(x,y) \leq \frac{b}{1+|x-y|^{2d+\delta}}$, *where* $b > 0$ *and* $\delta > 0$ *are constants. Let* $f(x)$ *be a bounded and integrable measurable function in* $\mathbb{R}^d$ *with respect to Lebesgue measure. Then for any* $T > 0$,

$$\sup_{l\geq 1; s,t\leq T} \Big|\frac{1}{l^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\frac{x}{l})f(\frac{y}{l})E\Big[(\lambda(x,\eta_s) - E\lambda(x,\eta_s))(\lambda(y,\eta_t) - E\lambda(y,\eta_t))\Big]dxdy\Big|$$

$$\leq w_d b_1 \|f\|(\frac{1}{d}+\frac{1}{\delta})\int_{\mathbb{R}^d}|f(y)|dy,$$

where b_1 *is a constant which does not depend on* f, *and* w_d *is the surface area of the unit ball in* $\mathbb{R}^d$.

Proof of Functional CLT

We can show that for any $t \geq 0$,

$$\sup_l E\Big[\frac{1}{l^{d/2}}\Big(\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f(\frac{x}{l})\eta_t(dx)\Big) - \frac{1}{l^{d/2}}\Big(\int_{\mathbb{R}^d} f_n(\frac{x}{l})\eta_t(dx) - E\int_{\mathbb{R}^d} f_n(\frac{x}{l})\eta_t(dx)\Big)\Big]^2 \to 0,$$

as $n \to \infty$, so the finite-dimensional distributions of $Y^{(l)}(f, \cdot)$ converge weakly to multivariate normal distributions.

Proof of Functional CLT

In order to show the relative compactness of $\{Y^{(l)}(f,\cdot) : l \geq 0\}$. We write

$$Y^{(l)}(f,t) = H_t^{(l)} + U_t^{(l)} - V_t^{(l)},$$

where

$$U_t^{(l)} = \tilde{U}_t^{(l)} - E\left[\tilde{U}_t^{(l)}\right], \quad V_t^{(l)} = \tilde{V}_t^{(l)} - E\left[\tilde{V}_t^{(l)}\right].$$

$$\tilde{U}_t^{(l)} = \frac{1}{l^{d/2}} \int_0^t \int_{\mathbb{R}^d \times [0,+\infty)^2} f(\frac{x}{l}) \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u) N(dx,ds,dr,du)$$

$$\tilde{V}_t^{(l)} = \frac{1}{l^{d/2}} \int_0^t \int_{\mathbb{R}^d \times [0,+\infty)^2} f(\frac{x}{l}) \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u) \mathbf{1}_{[0,t-s]}(r) N(dx,ds,dr,du)$$

$$= \frac{1}{l^{d/2}} \int_{\{(x,s,r,u): r\geq 0, s\geq 0, s+r\leq t\}} f(\frac{x}{l}) \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u) N(dx,ds,dr,du)$$

$$H_t^{(l)} = \frac{1}{l^{d/2}}\Big[\int_{\mathbb{R}^d\times[0,\infty)} f(\frac{x}{l})\mathbf{1}_{(t,\infty)}(r)\hat{\eta}_0(dx,dr) - E\int_{\mathbb{R}^d\times[0,\infty)} f(\frac{x}{l})\mathbf{1}_{(t,\infty)}(r)\hat{\eta}_0(dx,dr)\Big].$$

Note that $\int_0^t \int_{A\times[0,+\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)N(dx,ds,dr,du)$ denotes the total number of points which are born before and at time t in A,

$$\int_0^t \int_{A\times[0,+\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\mathbf{1}_{[0,t-s]}(r)N(dx,ds,dr,du)$$

denotes the total number of points which are not only born but also die before and at time t in A.

Lemma

Suppose λ is bounded by L and $a(x,y) \leq \frac{b}{1+|x-y|^{2d+\delta}}$, where $b > 0$ and $\delta > 0$ are constants. Then for each $T > 0$, there there exist two families $\{\gamma_l^{(i)}(\delta) : 0 < \delta < 1, l \geq 1\}, i = 1, 2$ of nonnegative random variables satisfying

$$E\left[(H_{t+h}^{(l)} - H_t^{(l)})^2 | \mathcal{F}_t\right] \leq E\left[\gamma_l^{(1)}(\delta) | \mathcal{F}_t\right],$$
$$E\left[(U_{t+h}^{(l)} - U_t^{(l)})^2 | \mathcal{F}_t\right] \leq E\left[\gamma_l^{(2)}(\delta) | \mathcal{F}_t\right],$$

for all $0 \leq t \leq T$, $0 \leq h \leq \delta$; in addition,

$$\lim_{\delta \to 0} \sup_l E\left[\gamma_l^{(i)}(\delta)\right] = 0$$

For any $x, y \in \mathbb{R}^d$, we define $q(x,y) = |x-y| \wedge 1$.

Lemma

Suppose λ is bounded by L and $a(x,y) \leq \frac{b}{1+|x-y|^{2d+\delta}}$, where $b > 0$ and $\delta > 0$ are constants. Fix $T > 0$. Then there there exists $C > 0$, such that

$$E\left[q^2(V_{t+h}^{(l)}, V_t^{(l)})q^2(V_t^{(l)}, V_{t-h}^{(l)})\right] \leq Ch^2$$

for all $0 \leq t \leq T+1$, $0 \leq h \leq t$. In addition,

$$\lim_{h\to 0} \sup_l E\left[(V_h^{(l)})^2\right] = 0.$$

The relative compactness of $H^{(l)}$, $U^{(l)}$ and $V^{(l)}$ can be prove by using the estimates in above lemmas and by Theorem 8.6, 8.8 in Chapter 3 of Ethier and Kurtz (1985). Then the relative compactness of $\{Y^{(l)}(f,\cdot) : l \geq 0\}$ follows the next lemma.

Lemma

Fix n. For $k = 1, \cdots, n$*, let* $\{Z^{(k,m)} : m = 1, 2, \cdots\}$ *be a sequence of stochastic processes whose paths are elements in* $D_{\mathbb{R}}[0, \infty)$ *endowed with the Skorohod topology. If for each* $1 \le k \le n$*,* $\{Z^{(k,m)}\}$ *is relatively compact in* $D_{\mathbb{R}}[0, \infty)$ *and all its limits have continuous paths a.s., then the sequence* $Z^{(m)} = (Z^{(1,m)}, \cdots, Z^{(n,m)})$ *is relatively compact in* $D_{\mathbb{R}^n}[0, \infty)$*, and hence* $Z^{(1,m)} + \cdots + Z^{(n,m)}$ *is relatively compact in* $D_{\mathbb{R}}[0, \infty)$*.*

Application to random packing problem

The following theorem follows immediately our main theorem for spatial pure birth process.

Theorem

For all $\tau \in (0, \infty)$ *and all* $d \geq 1$*, there exist constants* $0 < \sigma_{d,\tau} < \infty$ *such that*

$$n^{-1/2}\left[\eta_\tau(I_n A) - E\eta_\tau(I_n A)\right] \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \sigma_{d,\tau}^2)$$

and

$$n^{-1} Var\left[\eta_\tau(I_n A)\right] \to \sigma_{d,\tau}^2.$$

This theorem is slightly different from the result in Penrose and Yukich (2002).

References

- Garcia, N. L. and Kurtz, T. G. (2005). *Spatial birth and death processes as solutions of stochastic equations*. Alea 1, 281-303.
- Ethier, Stewart N. and Kurtz, Thomas G. (1985) *Markov Processes: Characterization And Convergence*. John Wiley and Sons Inc.
- McLeish, D. L. (1974), *Dependent central limit theorems and invariance principles*. Annals of Probability 2, 620-628.
- Mitoma, Itaru (1983), *Tightness of Probabilities On C*$([0,1];\psi')$ *and D*$([0,1];\psi')$, Annals of Probability 4, 989-999.
- Penrose, Mathew D.(2005) *Multivariate spatial central limit theorems with applications to percolation and spatial graphs*. Annals of Probability 33, 1945-1991.