On uniqueness

for Volterra-type stochastic equation

(Joint work with Tom Salisbury)

Volterra-type stochastic equation

$$X_t = X_0 + \int_0^t (t-s)^{-\alpha} \sigma(X_s) dB_s \, .$$

Parameters:
$\alpha \in (0, 1/2)$,
γ — Hölder exponent of σ; $\gamma < 1$.

Question: Pathwise Uniqueness?

Some other cases:

- $\gamma = 1$. Lipschitz case. PU follows easily.

- $\alpha = 0$. PU for $\gamma \geq 1/2$
  by Yamada-Watanabe (71).

Motivation

Consider the SPDE

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t),$$

or precisely

$$\begin{aligned} X(t,x) \;&=\; \int p_t(x-y)X(0,y)dy \\ &\quad + \int_0^t \int p_{t-s}(x-y)\sigma(X(s,y))\,W(dy,ds), \end{aligned}$$

where $\dot{W}$ is a space-time white noise.

Existence: function-valued solution exists if $d=1$.

Uniqueness?

Pathwise uniqueness (PU):
X^1, X^2 — two solutions, $X^1(0,\cdot) = X^2(0,\cdot)$
$\Longrightarrow X^1(t,\cdot) = X^2(t,\cdot)\,, \forall t > 0$.

σ — Lipschitz $\Longrightarrow$ PU follows easily.

σ - non-Lipschitz: ?

Let $\sigma(x)$ be Hölder continuous with exponent γ.
Ongoing work with E. Perkins:
$\gamma \geq 0.95$ — **PU** for

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where $\dot{W}$ is space-time white noise.
Open: critical γ_0 such that
$\gamma > \gamma_0$ — **PU**,
$\gamma < \gamma_0$ — no **PU**.
We want to consider equations that are close to the above.
One way is to take less singular (spatially) noise.
The noise $\dot{W}$ is "white" in time and (possibly) "colored" in space, that is,

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)k(x-y).$$

Assumptions $k(z) \leq |z|^{-\alpha}$, $0 \leq \alpha < d$.
$\sigma(x)$ is Hölder continuous with exponent γ.

Existence of function-valued solution:
$0 \leq \alpha < 2 \wedge d$, Peszat-Zabczyk(00), Dalang(99) (for Lipschitz case. Similar for non-Lipschitz).

Theorem 1 (Sturm, Perkins, M., 05)
PU holds if

$$\alpha < 2\gamma - 1.$$

Super-Brownian motion

Branching Brownian motions.

$\mathbf{R}^d$
X^n:
$\sim n$ particles in $\mathbf{R}^d$ at time 0.
$\frac{1}{n}, \frac{2}{n}, \ldots$ — times of death or split,
$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.
Critical branching:
mean number of offspring = 1.

New particles move as independent Brownian motions.

$$X_t^n(A) = \frac{\#\text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbf{R^d}.$$

$X^n \Rightarrow X$,
X is a super-Brownian motion — measure-valued process.

Properties

Singular measure if $d > 1$.

Existence of density only in $d = 1$:

$X_t(dx) = X_t(x)dx$

$\mathbf{d = 1}$. $X_t(x)$ is jointly continuous in (t, x). N. Konno, T. Shiga(88); M. Reimers (89):

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

$\dot{W}$ — Gaussian space-time white noise.

Pathwise uniqueness (PU) for the above SPDE is an open question. ($\sqrt{X}$ is non-Lipschitz.) Numerous attempts to prove **PU** failed.

Weak uniqueness (in law) holds by duality argument.

Consider more general super-Brownian motion:

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2}\Delta X_t(x) + \sqrt{\lambda_s(x) X_t(x)}\dot{W}, \quad t \geq 0,\ x \in R.($$

$\lambda_s(x)$ is a "rate" of branching at the point x at time s.

Replace $\lambda_s(x)dx$ by a singular measure $\rho_s(dx)$. X is a catalytic SBM with catalyst ρ (studied by Dawson, Fleischmann, Delmas and others).

Particular case: $\rho = \delta_0$ — point catalyst.

Particular case: $\rho = \delta_0$ — point catalyst.

Let l_t^0 be a local time, that is

$$l_t^0 = \int_0^t \int_R \delta_0(y) X_s(dy)\, ds.$$

Then SBM with a point catalyst at $x = 0$ satisfies the martingale problem:

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), M_0(\phi) = 0 \\ X_t(\phi) = X_0(\phi) + \int_0^t X_s(\Delta\phi/2)\, ds + M_t(\phi), \\ \text{where } M_t(\phi) \text{ is a continuous } \mathcal{F}_t\text{-martingale} \\ \text{ and } \langle M(\phi)\rangle_t = \phi(0)^2 l^0(t). \end{cases} \quad (2)$$

Pretend for a second that $l^0(ds)$ is absolutely continuous, that is,

$$l_t^0 = \int_0^t X_s(0)\, ds, \quad (3)$$

Then $X_t(\cdot)$ solves a degenerate SPDE

$$X_t(x) = \int_{\mathbb{R}} p_t(x-y) X_0(dy) + \int_0^t p_{t-s}(x) \sqrt{X_s(0)} dB_s\,,$$

where $p_t(x)$ is a transition density of Brownian motion.

Set $x = 0$ to get the following SDE

$$X_t(0) = \int_{\mathbb{R}} p_t(y) X_0(dy) + c \int_0^t (t-s)^{-1/2} \sqrt{X_s(0)} dB_s \,. \quad (4)$$

$X_0 =$ const. Volterra equation?

However (3) is wrong—the local time $l^0(ds)$ is singular
$X_t(dx)$ does not have a density at $x = 0$.
No solution to (4).

Instead of 1/2 take α:

Easy to get *existence* of solution to the following SDE

$$X_t(0) = X_0 + c \int_0^t (t-s)^{-\alpha} \sqrt{X_s(0)} dB_s \,. \quad (5)$$

for $0 \le \alpha < 1/2$.

Uniqueness?
Let σ be a Hölder continuous with exponent γ.

We will consider uniqueness problem the following Volterra-type stochastic equation

$$X_t = X_0 + \int_0^t (t-s)^{-\alpha} \sigma(X_s) dB_s \,. \quad (6)$$

Recall: pathwise uniqueness for SDEs

$$dX_t = \sigma(X_t)dB_t$$

B_t is a one-dimensional Brownian motion.

Theorem 2 (Yamada, Watanabe (71)) *If σ is Hölder continuous with exponent 1/2, then PU holds.*

Remark 1 *There are counter examples for σ which is Hölder continuous with exponent less than 1/2.*

Proof of Theorem 2 Define (in a special way) function $\phi_n \in C_c^\infty(R)$ s.t.

$$\begin{aligned} \phi_n(x) &\uparrow |x|, \text{ as } n \to \infty, \\ \phi_n'' &\to \delta_0\,, \text{ as } n \to \infty. \end{aligned}$$

Define $\tilde{X} = X^1 - X^2$. Hence

$$\begin{aligned} d\tilde{X}_t &= (\sigma(X_t^1) - \sigma(X_t^1))dB_t \\ \tilde{X}_0 &= 0. \end{aligned}$$

Ito's formula:

$$\begin{aligned} \phi_n(\tilde{X}_t) &= \int_0^t \phi_n'(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2))\, dB_s \\ &\quad + \frac{1}{2}\int_0^t \phi_n''(\tilde{X}_s)(\sigma(X_s^1) - \sigma(X_s^2))^2\, ds \end{aligned}$$

By the special choice of the function ϕ_n and Hölder assumptions on σ one can show

$$\begin{aligned} E\left[\phi_n(\tilde{X}_t)\right] &\leq cE\left[\int_0^t \phi_n''(\tilde{X}_s)|\tilde{X}_s|\, ds\right] \\ &\to 0, \text{ as } n \to \infty. \end{aligned}$$

Proof of Theorem 1 Recall

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where σ is Hölder continuous with exponent γ and $\dot{W}$ is a colored noise

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)k(x-y).$$

with

$$k(z) \le |z|^{-\alpha},\ \alpha > 0.$$

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \tilde{X}_t(x) + (\sigma(X_t^1(x)) - \sigma(X_t^2(x)))\dot{W}(t,x).$$

Take again the functions ϕ_n:

$$\phi_n(x) \uparrow |x|, \text{ as } n \to \infty,$$
$$\phi_n'' \to \delta_0 \text{ as } n \to \infty.$$

Denote

$$\tilde{\sigma}(s,x) \equiv \sigma(X_s^1(x)) - \sigma(X_s^2(x)).$$

Ito:

$$E\left[\phi_n(\tilde{X}_t(f))\right] = E\left[\int_0^t \phi_n'(\tilde{X}_s(f))\tilde{X}_s(\frac{1}{2}\Delta f)\,ds\right]$$
$$+ E\left[\frac{1}{2}\int_0^t \int_{R^{2d}} \phi_n''(|\tilde{X}_s(f)|)\tilde{\sigma}(s,z)\tilde{\sigma}(s,y)f(z)f(y)k(z-y)dz\,dy\,ds\right]$$

Let $f = f_x^n \to \delta_x$.

Also $\mathbf{H}(\gamma) \Longrightarrow |\tilde{\sigma}(s,x)| \le c|\tilde{X}_s(x)|^\gamma$ and hence

$$E\left[\phi_n(\tilde{X}_t(f_x^n))\right] \le E\left[\int_0^t \phi_n'(\tilde{X}_s(f_x^n))\tilde{X}_s(\frac{1}{2}\Delta f_x^n)\,ds\right]$$
$$+ E\left[\frac{1}{2}\int_0^t \int_{R^{2d}} \phi_n''(|\tilde{X}_s(f_x^n)|)|\tilde{X}_s(z)|^\gamma |\tilde{X}_s(y)|^\gamma \right.$$
$$\left. \times f_x^n(z) f_x^n(y) k(z-y) dz\,dy\,ds\right]$$
$$= I^{1,n} + I^{2,n}.$$

"Easy" to check:

$$\limsup_{n\to\infty} I^{1,n} \le \int_0^t \frac{1}{2}\Delta E\left[\left|\tilde{X}_s(x)\right|\right] ds.$$

Crucial for $I^{2,n}$: Hölder exponent of $\tilde{X}$.

Suppose $\tilde{X}$ is Hölder continuous with exponent ξ.

Then we can show:

$$I^{2,n} \to 0, \text{ if } \alpha < \xi(2\gamma - 1).$$

$$\Longrightarrow E\left[|\tilde{X}_t(x)|\right] \le \int_0^t \frac{1}{2}\Delta E\left[\left|\tilde{X}_s(x)\right|\right] ds$$

$$\Longrightarrow E\left[|\tilde{X}_t(x)|\right] = 0, \text{ if } \alpha < \xi(2\gamma - 1).$$

We got condition for PU:

$$\alpha < \xi(2\gamma - 1).$$

Proposition 2 (Sanz-Solé, Sarrà) *For any* $\xi < 1 - \frac{\alpha}{2}$*,* $\tilde{X}_s(\cdot)$ *is Hölder continuous with exponent* ξ*.*

By Theorem of Sanz-Solé, Sarrà we get

$$\alpha < \frac{2\gamma - 1}{\gamma + 1/2}.$$

Bad: $\gamma \nearrow 1 \Longrightarrow \alpha < 2/3.$

Proposition 3 (Sturm, Perkins, M.)
At the points x where $\tilde{X}_s(x) = 0$, $\tilde{X}_s(\cdot)$ is ξ-Hölder continuous $\forall \xi < 1$.

Remark Mueller-Tribe have the result similar to Proposition 3.

By condition on PU ($\alpha < \xi(2\gamma - 1)$) we get

$$\alpha < 2\gamma - 1$$

and this finishes the proof of Theorem 1.

Volterra-type stochastic equation

$$X_t = X_0 + \int_0^t (t-s)^{-\alpha} \sigma(X_s) dB_s \,.$$

$\alpha \in (0, 1/2)$,
$\gamma \in (1/2, 1]$ — Hölder exponent of σ.

Theorem 3 *Let*

$$\gamma > \frac{1}{2(1-\alpha)}.$$

Then **PU** *holds.*

$$X_t = X_0 + \int_0^t (t-s)^{-\alpha}\sigma(X_s)dB_s\,.$$

Idea: represent it as a solution to SPDE.

Fix θ such that $\frac{1}{2+\theta} = \alpha$. Define

$$\Delta_\theta \;=\; \frac{2}{(2+\theta)^2}\frac{\partial}{\partial x}\,|x|^{-\theta}\,\frac{\partial}{\partial x} \tag{7}$$

Then the function

$$p^\theta(t,x) = \frac{C_\theta}{t^{\frac{1}{2+\theta}}}e^{-\frac{|x|^{2+\theta}}{2t}}$$

is a solution to

$$\begin{cases} \frac{\partial u}{\partial t} &= \Delta_\theta u \\ u_0 &= \delta_0 \end{cases} \tag{8}$$

Let X be a solution to the following SPDE on $R_+ \times R$

$$\frac{\partial X(t,x)}{\partial t} = \Delta_\theta X(t,x) + \delta_0(x)\sigma(X(t,x))\dot{B}(t). \quad (9)$$

More precisely X solves

$$X(t,x) = S_t X_0(x) + \int_0^t \int_{\mathbb{R}} p^\theta(t-s,x)\sigma(X(s,0))dB(s)$$

where $S_t\,, t \geq 0$ is the semigroup generated by Δ_θ.

Existence: easy by tightness argument.

Let $X_0(\cdot) = x_0 =$ const. Then

$$X(t,x) = X_0 + \int_0^t p^\theta(t-s,x)\sigma(X(s,0))dB(s)$$

In particular for $x = 0$ we have

$$X(t,0) = x_0 + \int_0^t c_\theta(t-s)^{-\alpha}\sigma(X(s,0))dB(s). (10)$$

Remark 4 *Uniqueness for (10) follows from uniqueness for (9).*

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \Delta_\theta \tilde{X}_t(x) + \delta_0(x)(\sigma(X_t^1(x)) - \sigma(X_t^2(x)))\dot{B}(t).$$

Take again the functions ϕ_n:

$$\phi_n(x) \uparrow |x|, \text{ as } n \to \infty,$$
$$\phi_n'' \to \delta_0 \text{ as } n \to \infty.$$

Denote

$$\tilde{\sigma}(s,x) \equiv \sigma(X_s^1(x)) - \sigma(X_s^2(x)).$$

Ito:

$$E\left[\phi_n(\tilde{X}_t(f))\right] = E\left[\int_0^t \phi_n'(\tilde{X}_s(f))\tilde{X}_s(\Delta_\theta f)\,ds\right]$$
$$+ E\left[\frac{1}{2}\int_0^t \phi_n''(|\tilde{X}_s(f)|)\tilde{\sigma}(s,0)^2 f(0)^2 ds\right].$$

Let $f = f_x^n \to \delta_x$.

$$E\left[\phi_n(\tilde{X}_t(f_x^n))\right] = E\left[\int_0^t \phi_n'(\tilde{X}_s(f_x^n))\tilde{X}_s(\Delta_\theta f_x^n)\, ds\right]$$
$$+ E\left[\frac{1}{2}\int_0^t \phi_n''(|\tilde{X}_s(f_x^n)|)\tilde{\sigma}(s,0)^2 f_x^n(0)^2\, ds\right]$$
$$= \quad I^{1,n} + I^{2,n}.$$

Again

$$\limsup_{n\to\infty} I^{1,n} \le \int_0^t \Delta_\theta E\left[\left|\tilde{X}_s(x)\right|\right]\, ds.$$

For $I^{2,n}$ crucial:
Regularity of $\tilde{X}_s(x)$ for x close to 0.
Suppose for all $|x|$ small:

$$\left|\tilde{X}_s(x)\right| \le |x|^\xi$$

Then we can show:

$$I^{2,n} \to 0, \text{ if } \gamma > \frac{1}{2} + \frac{1}{2\xi}.$$

$$\implies E\left[|\tilde{X}_t(x)|\right] \le \int_0^t \Delta_\theta E\left[\left|\tilde{X}_s(x)\right|\right]\, ds$$

$$\implies E\left[|\tilde{X}_t(x)|\right] = 0, \text{ if } \gamma > \frac{1}{2} + \frac{1}{2\xi}.$$

We got condition for PU:

$$\gamma > \frac{1}{2} + \frac{1}{2\xi}.$$

Proposition 5

At times s where $\tilde{X}_s(0) = 0$, we have

$$\left|\tilde{X}_s(x)\right| \leq C\,|x|^{\xi}\,,$$

for any

$$\xi < \frac{1}{\alpha}\left(\frac{1/2-\alpha}{1-\gamma} \wedge 1\right)$$

By condition on PU we get

$$\alpha < \frac{\gamma - 1/2}{\gamma}$$

and this finishes the proof of Theorem 3.

Proof of Proposition 5

$$
\begin{aligned}
\tilde{X}_{t'}(0) - \tilde{X}_t(0) &= \int_0^t ((t'-s)^{-\alpha} - (t-s)^{-\alpha})\tilde{\sigma}(s,0)\, dB_s \\
&\quad + \int_t^{t'} (t'-s)^{-\alpha}\tilde{\sigma}(s,0)\, dB_s\,.
\end{aligned}
$$

We assume that $\tilde{X}(t,0) = 0$, and $\tilde{X}(t,0)$ is Hölder with exponent η. Then

$$
|\tilde{\sigma}(s,o)| \le |\tilde{X}(s,0)|^{\gamma} \le c(t-s)^{\eta\gamma}.
$$

Formally

$$
\begin{aligned}
|\tilde{X}_{t'}(0) - \tilde{X}_t(0)| &\le c\sqrt{\int_0^t ((t'-s)^{-\alpha} - (t-s)^{-\alpha})^2 \tilde{\sigma}(s,0)^2\, ds} \\
&\quad + c\sqrt{\int_t^{t'} (t'-s)^{-2\alpha}\tilde{\sigma}(s,0)^2\, ds} \\
&\le c\sqrt{\int_0^t ((t'-s)^{-\alpha} - (t-s)^{-\alpha})^2 (t-s)^{2\gamma\eta}\, ds} \\
&\quad + c\sqrt{\int_t^{t'} (t'-s)^{-2\alpha}(t-s)^{2\gamma\eta}\, ds} \\
&\le c|t'-t|^{(1/2-\alpha+\eta\gamma)\wedge 1}
\end{aligned}
$$

Iterate to get

$$
|\tilde{X}_{t'}(0) - \tilde{X}_t(0)| \le c|t'-t|^{\eta}
$$

for any

$$
\eta < \frac{1/2-\alpha}{1-\gamma} \wedge 1.
$$

Similarly

$$
\begin{aligned}
|\tilde{X}_t(x) - \tilde{X}_t(0)| &= |\int_0^t (p^\theta_{t-s}(x) - p^\theta_{t-s})\tilde{\sigma}(s,0)\,dB_s| \\
&\leq c\sqrt{\int_0^t (p^\theta_{t-s}(x) - p^\theta_{t-s}(0)^2 (t-s)^{2\gamma\eta}\,ds} \\
&\quad \cdots \\
&\leq |x|^{\eta/\alpha}
\end{aligned}
$$

for any

$$\eta < \frac{1/2 - \alpha}{1 - \gamma} \wedge 1.$$

Theorem 4 *For any $\alpha < 1/2$, weak uniqueness holds for non-negative solutions of*

$$X_t = X_0 + \int_0^t (t-s)^{-\alpha} \sqrt{|X_s|} dB_s \,.$$

Proof Duality argument.