Spectral gap and convex concentration inequalities for birth-death processes

Yutao Ma

University Paris 10 and Beijing Normal University

Joint work with Wei Liu

The fifth workshop on Markov processes and related topics

July 14-18 2007, Beijing

Outline

Birth-death processes

Let $(X_t)_{t\geq 0}$ be a birth-death process on $\mathbb{N} = \{0, 1, 2, \cdots\}$ with generator $\mathcal{L}$ satisfying for any function $f : \mathbb{N} \mapsto \mathbb{R}$,

$$\mathcal{L}f(i) = b_i(f(i+1) - f(i)) + a_i(f(i-1) - f(i)),$$

where $b_i > 0\,(i \geq 0)$, $\quad a_i > 0\,(i > 0)$ and $a_0 = 0$. Define a measure

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 b_1 \cdots b_{n-1}}{a_1 a_2 \cdots a_n}, \quad n \geq 1,$$

then μ is an invariant measure of $\mathcal{L}$. We suppose always that

$$\sum_{n\geq 0} \mu_n \sum_{i\geq n} (\mu_i b_i)^{-1} = \infty, \quad \text{and} \quad C := \sum_{n=0}^{+\infty} \mu_n < +\infty.$$

Denote by $\pi_n = \frac{\mu_n}{C}$, $n \geq 0$ the reversible invariant probability.

Spectral gap

Let λ_1 be the spectral gap of $\mathcal{L}$ in $L^2(\pi)$, i.e., the infimum of the spectrum of the operator $-\mathcal{L}$ in $L^2(\pi)$. P_t is the semigroup of the process. On one hand, λ_1 is the best constant ε in

$$||P_t f - \pi(f)||_{L^2(\pi)} \leq e^{-\varepsilon t}||f - \pi(f)||_{L^2(\pi)}.$$

On the other hand,

$$\lambda_1 = \inf\{\mathcal{E}_\pi(f), \pi(f) = 0, \pi(f^2) = 1\}$$

and λ_1 is the optimal constant C in the following Poincaré inequality:

$$C\, Var_\pi(f) \leq \mathcal{E}_\pi(f),$$

where $Var_\pi(f)$ and $\mathcal{E}_\pi(f)$ are respectively the variance and Dirichlet form of f with respect to π.

Coupling method

- In 1993, Chen and Wang used the coupling method to obtain the first eigenvalue on manifold.
- In 1996, Chen proved two exact variational formulas of the spectral gap for birth-death processes by coupling.
- The diffusion case is due to Chen and Wang in 1997.

Strategy: find some appropriate metric d so that

$$\|P_t f\|_{\mathrm{Lip}(d)} \le e^{-\varepsilon t}\|f\|_{\mathrm{Lip}(d)}$$

with $\varepsilon > 0$ as large as possible.

Other approaches

- Chen gave an analytic approach for both birth-death and diffusion processes in 1999.
- Miclo, in 1999, extended the Muckenhoupt's generalized Hardy inequality from $\mathbb{R}$ to $\mathbb{N}$ and so derived upper and lower bounds on λ_1 which are different only by a factor 4.
- In 2000, Chen obtained bounds on spectral gap with variational formulas which recovered the results of Miclo.

Direct motivation

In 2007, Djellout and Wu considered one dimensional diffusion processes with generator $\mathcal{L}$, and obtained an explicit representation of $(-\mathcal{L})^{-1}$ on some Lipschitz space, so yields another proof of Chen-Wang's variational formula of λ_1.

Key points: Lipschitz space; Poisson equation, Spectral theory.

Lipschitz norm

Given ρ an increasing function on $\mathbb{N}$, $d_\rho(i,j) := |\rho(i) - \rho(j)|$.

f is ρ-Lipschitz if

$$||f||_{\mathrm{Lip}(\rho)} := \sup_{i\neq j} \frac{|f(i)-f(j)|}{d_\rho(i,j)} < \infty,$$

which is equivalent to

$$||f||_{\mathrm{Lip}(\rho)} = \sup_{i\geq 0} \frac{|f(i+1)-f(i)|}{d_\rho(i,i+1)} < \infty.$$

Denote by $(C^0_{\mathrm{Lip}(\rho)}, ||\cdot||_{\mathrm{Lip}(\rho)})$ the space of Lipschitz functions with respect to ρ with zero mean under π.

Is $(-\mathcal{L})^{-1}$ well defined on $C^0_{\mathrm{Lip}(\rho)}$?

Yes! The equation $\mathcal{L}f = 0$ admits constant solutions, then an unique identically zero solution when $\pi(f) = 0$ is required. So for $g \in C^0_{\mathrm{Lip}(\rho)}$, there exists one and only one solution f on $\mathbb{N}$ with $\pi(f) = 0$ to the Poisson equation

$$-\mathcal{L}f = g.$$

Equivalently $(-\mathcal{L})^{-1}$ is well defined on $C^0_{\mathrm{Lip}(\rho)}$.

Poisson equation

Consider the Poisson equation $-\mathcal{L}f = g$ with $\pi(g) = 0$. Then we have

$$f(i+1) - f(i) = -\frac{\sum_{j=0}^{i} \pi_j g(j)}{\pi_{i+1} a_{i+1}} = \frac{\sum_{j=i+1}^{\infty} \pi_j g(j)}{\pi_{i+1} a_{i+1}}. \tag{1}$$

Representation of $||(-\mathcal{L})^{-1}||_{\mathrm{Lip}(\rho)}$

Theorem 1: Suppose that $\rho \in L^1(\pi)$, we have

$$||(-\mathcal{L})^{-1}||_{\mathrm{Lip}(\rho)} = \sup_{i\geq 0} \frac{\sum_{j=i+1}^{\infty} \pi_j(\rho(j) - \pi(\rho))}{\pi_{i+1} a_{i+1}(\rho(i+1) - \rho(i))} =: I(\rho). \qquad (2)$$

Remarks

- Suppose that the function γ satisfies $-\mathcal{L}\gamma = \rho - \pi(\rho)$ with $\pi(\gamma) = 0$. Then $||\gamma||_{\mathrm{Lip}(\rho)} = I(\rho)$.
- When $I(\rho)$ is finite, then the operator $(-\mathcal{L})^{-1}$ maps $C^0_{\mathrm{Lip}(\rho)}$ to $C^0_{\mathrm{Lip}(\rho)}$ itself. Otherwise, at least γ is not ρ-Lipschitz.

Sketch of Proof

Given g a function satisfying $\pi(g) = 0$ and $||g||_{\mathrm{Lip}(\rho)} = 1$, we have for any $k \geq 0$,

$$\sum_{i \geq k} \pi_i g(i) \leq \sum_{i \geq k} \pi_i(\rho(i) - \pi(\rho)). \tag{3}$$

Now consider two Poisson equations

$$-\mathcal{L}f = g \quad \text{and} \quad -\mathcal{L}\gamma = \rho - \pi(\rho).$$

By the expression (1) and the inequality (3), we have a comparison relation

$$||f||_{\mathrm{Lip}(\rho)} \leq ||\gamma||_{\mathrm{Lip}(\rho)}.$$

Moreover, $||\gamma||_{\mathrm{Lip}(\rho)} = I(\rho)$, the supremum is attained.

Variational formula of spectral gap

Theorem 2: Let λ_1 be the spectral gap of $\mathcal{L}$ in $L^2(\pi)$, we have

$$\lambda_1 = \sup_{\rho \in \mathcal{A}} I(\rho)^{-1},$$

where $\mathcal{A} = \{\rho : \rho \in L^1(\pi), \rho \text{ is increasing } \}.$

- $\lambda_1 \geq \sup_{\rho\in\mathcal{A}} I(\rho)^{-1}$ is derived from the following two points:
  - The exact expression of $I(\rho)$;
  - $-\mathcal{L}$ is nonnegative definite and self-adjoint.
- The equality is achieved with $\bar{\rho}$ the eigenfunction of $-\mathcal{L}$ corresponding to λ_1 in the weak sense, i.e., $-\mathcal{L}\bar{\rho} = \lambda_1\bar{\rho}$.

Definition

Two random variables F and G satisfy a convex concentration inequality (CCI in brief) if

$$\mathbb{E}[\phi(F)] \leq \mathbb{E}[\phi(G)] \tag{4}$$

for all convex functions $\phi : \mathbb{R} \mapsto \mathbb{R}$.

- 1963, Hoeffding
- 2003, Klein
- 2006, Klein, Ma, Privault

Taking $\phi(x) = e^{\lambda x}$, we have for any $x > 0$,

$$\mathbb{P}(F \geq x) \leq \inf_{\lambda > 0} \mathbb{E}[e^{\lambda(F-x)}] \leq \inf_{\lambda > 0} \mathbb{E}[e^{\lambda(G-x)}].$$

Our aim

Consider $S_t = \int_0^t g(X_s)ds$ with $\pi(g) = 0$. We want to prove CCI for $(S_t)_{t\geq 0}$, then we could have a deviation inequality for $t^{-1}S_t$, which characterizes the decay of the empirical measure $L_t := t^{-1}\int_0^t \delta_{X_s}ds$ to π.

CCI for pure jump martingales

Theorem A : Let $(M_t)_{t\geq 0}$ be a pure jump martingale on some probability space $(E, \mathcal{E}, \mathbb{P})$ satisfying for all $t \geq 0$,

$$|\Delta M_t| \leq K \quad \text{and} \quad || < M >_t ||_\infty < +\infty,$$

where $|| < M >_t ||_\infty = ess\sup_{\omega \in E} | < M >_t (\omega)|$. Then for any function $\phi \in \mathcal{C}_c$ and any $t \geq 0$, we have

$$\mathbb{E}_\mathbb{P}\Big[\phi\Big(M_t - \mathbb{E}_\mathbb{P}[M_t]\Big)\Big] \leq \mathbb{E}\Big[\phi\Big(KN_{||<M>_t||_\infty/K^2} - || < M >_t ||_\infty / K\Big)\Big], \tag{5}$$

where $\mathcal{C}_c := \{\phi, \phi$ convex with ϕ'' non-decreasing $\}$ and N is Poisson process. Moreover, (5) still holds if $|| < M >_t ||_\infty$ is replaced by some deterministic function bounding above such a quantity.

Lyons-Zheng's decomposition

Consider again the Poisson equation $-\mathcal{L}f = g$. Define $\forall 0 \le t \le T$,

- $M_t \triangleq f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$
- $\widetilde{M}_t \triangleq f(X_0) - f(X_t) - \int_0^t \mathcal{L}f(X_s)ds.$

Obviously $S_t = \frac{M_t + \widetilde{M}_t}{2}$. Since the process is reversible with respect to π, M_t and $\widetilde{M}_t$ have the same distribution with respect to $\mathbb{P}_\pi$. Then by the convexity of ϕ and Jessen's inequality, we have

$$\mathbb{E}_\pi[\phi(S_t)] \le \mathbb{E}_\pi[\phi(\frac{M_t + \tilde{M}_t}{2})] \le \mathbb{E}_\pi[\phi(M_t)].$$

$(M_t)_{t \ge 0}$ is a pure jump martingale and furthermore we have

- $|\Delta M_t| \le \sup_{k \ge 1} |f(k) - f(k-1)|,$
- $< M >_t = \int_0^t \Gamma(f)(X_s)ds.$

Essential Assumptions

- Assumption A:

$$K := \sup_{k \geq 1} \frac{|\sum_{i=0}^{k-1} \pi_j g(j)|}{\pi_k a_k} < +\infty.$$

- Assumption B:

$$\sup_{k \geq 1} \left\{ a_k \left(\frac{\sum_{i=0}^{k-1} \pi_j g(j)}{\pi_k a_k} \right)^2 + b_k \left(\frac{\sum_{i=0}^{k} \pi_j g(j)}{\pi_k a_k} \right)^2 \right\} < \infty.$$

Remark: $K = \sup_{k \geq 1} |f(k) - f(k-1)|$ and the Assumption B is equivalent to $||\Gamma(f)||_\infty < \infty$.

Theorem 3: Provided that the Assumptions *A* and *B* are satisfied, we have for any function $\phi \in \mathcal{C}_c$,

$$\mathbb{E}_\pi\left[\phi\left(\int_0^t g(X_s)ds\right)\right] \le \mathbb{E}\left[\phi\left(KN_{||\Gamma(f)||_\infty t/K^2} - ||\Gamma(f)||_\infty t/K\right)\right]. \quad (6)$$

Remark: Suppose that $\rho - \pi(\rho)$ satisfies the Assumptions *A* and *B*, (6) holds for any $g \in C^0_{\mathrm{Lip}(\rho)}$.

Remark 1. Suppose g satisfies the Assumptions A and B,

$$\mathbb{P}_\pi\left(t^{-1}\int_0^t g(X_s)ds \geq x\right) \leq \exp\left\{-\frac{||\Gamma(f)||_\infty t}{K^2} h\left(\frac{Ky}{||\Gamma(f)||_\infty}\right)\right\}, \qquad (7)$$

where $h(u) = (1+u)\log(1+u) - u$. Furthermore, if $\rho - \pi(\rho)$ satisfies the Assumptions A and B, we have for any ρ-Lipschitz function g, (7) is correct.

Remark 2. Suppose that ν is a probability on $\mathbb{N}$ absolutely continuous with respect to π with $\frac{d\nu}{d\pi} \in L^2(\pi)$ and $g - \pi(g)$ verifies the Assumptions A and B. With the inequality (7) and Cauchy-Schwarz inequality, we could have for any $x > 0$ and $t \geq 0$,

$$\mathbb{P}_\nu\left(t^{-1}\int_0^t g(X_s)ds - \pi(g) \geq x\right) \leq \|\frac{d\nu}{d\pi}\|_{L^2(\pi)} \exp\left\{-\frac{\|\Gamma(f)\|_\infty t}{2K^2} h\left(\frac{Kx}{\|\Gamma(f)\|_\infty}\right)\right\}.$$

Example 1: $M/M/1$ queueing process

- The corresponding generator:

$$\mathcal{L}f(i) = \lambda(f(i+1) - f(i)) + \nu 1_{i\neq 0}(f(i-1) - f(i)).$$

- $\sigma := \frac{\lambda}{\nu} < 1$, the invariant probability π, $\quad \pi_k = \sigma^k(1-\sigma), \forall k \in \mathbb{N}$.

Corollary: Let ρ be an increasing function in $L^1(\pi)$. Suppose that

$$\sup_{k\geq 1} \sum_{i=0}^{\infty} \pi_i \big(\rho(i+k) - \rho(i)\big) < \infty,$$

then the Assumptions A and B are satisfied for any $g \in C^0_{\mathrm{Lip}(\rho)}$.

Example 2: $M/M/\infty$ queueing process

- The generator

$$\mathcal{L}f(i) = \lambda(f(i+1) - f(i)) + \nu i(f(i-1) - f(i)).$$

- The invariant probability π : $\quad \pi_k = e^{-\sigma}\frac{\sigma^k}{k!}, \quad k \in \mathbb{N}$ with $\sigma := \lambda/\nu$.

Corollary: Let $\rho(k) = \sqrt{k}, \forall k \geq 1$. Then the Assumptions A and B are satisfied for any $g \in C^0_{\mathrm{Lip}(\rho)}$.

Thank you for your attention