Catalytic Discrete State Branching Models and Related Limit Theorems

Chunhua Ma

Beijing Normal University

(A joint work with Zenghu Li)

0.Introduction; catalytic CBI-processes

- Dawson-Fleishmann ('97) introduced catalytic branching processes in the measure-valued setting.

- Modeling of catalytic reactions.

  the reactant only branches in the presence of the catalyst.

- Dawson-Li ('06) defined catalytic CBI-processes (without spatial motion) by a system of stochastic integral equations.

  the catalyst process is a CBI-process.

  the reactant process is a 'CBI-process' with random branching rate proportional to the catalyst.

An example of catalytic CBI-processes

● For real constants σ_1, σ_2, and two one-dimensional Brownain motions $B_1(\cdot)$ and $B_2(\cdot)$

$$dX(t) = \sigma_1\sqrt{X(t)}dB_1(t), \quad (1)$$

$$dY(t) = \sigma_2\sqrt{X(t)Y(t)}dB_2(t). \quad (2)$$

$B_1(\cdot)$ and $B_2(\cdot)$ may not be independent!

● From the modeling perspective, the catalyst and the reactant may involve dependent mechanisms.

● Duffie *et al.* ('03): affine processes.

Dawson-Li ('06): regular affine processes arise in some fluctuation limit theorems of catalytic CBI-processes.

Our questions.

- To find the discrete state counterpart: catalytic DBI-processes.

  the 'discrete particle' picture may be a guide to our intuition

- Diffusion approximations.

catalytic DBI-processes	$\xrightarrow{\textit{really}\,?}$	catalytic CBI-processes

- Li ('00): DBI-processes $\xrightarrow{fluctuate}$ OU-processes.

catalytic DBI-processes	$\xrightarrow{\textit{how to rescale}\,?}$	Affine markov processes

A discrete-state version of Dawson-Li ('06).

1. Catalytic discrete state branching processes

- A discrete state branching process (DB) is an $\mathbb{N}$-valued Markov chain $\{\xi(t)\}$ with Q-matrix of the form

$$q_{ij} = \begin{cases} l_1 i p_{j-i+1} & \text{if } j \geq i-1 \text{ and } j \neq i, \\ l_1 i (p_1 - 1) & \text{if } j = i, \\ 0 & \text{others}, \end{cases} \tag{3}$$

where $l_1 > 0$, $\{p_i : i = 0, \cdots\}$ a discrete distribution on $\mathbb{N}$.

- Let $\xi(0) \in \mathbb{N}$. A realization of $\{\xi(t)\}$ is given by

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{N}} \int_0^{l_1 \xi(s-)} (z-1) J_1(ds, dz, du), \tag{4}$$

where $J_1(ds, dz, du)$ a Poisson r.m. on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $ds \mu_1(dz) du$ and $\mu_1(\{i\}) = p_i$.

- Let $l_2 > 0$ and let μ_2 be another probability on $\mathbb{N}$. Suppose that $J_2(ds, dz, du)$ is another Poisson r.m. on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $ds\mu_2(dz)du$.

- Given any $\eta(0) \in \mathbb{N}$, define another process $\eta(t)$ by

$$\eta(t) = \eta(0) + \int_0^t \int_{\mathbb{N}} \int_0^{l_2\xi(s-)\eta(s-)} (z-1)J_2(ds, dz, du). \quad (5)$$

Intuitively, $\eta(t)$ is a branching process with branching rate $l_2\xi(s-)$ at time $s \geq 0$. Then we call $(\xi(t), \eta(t))$ a catalytic branching system or catalytic DB-process, where $\xi(t)$ is the catalyst and $\eta(t)$ is the reactant.

Reformulation of catalytic branching processes

- Let $N_1(ds, dz, du)$ be a Poisson r.m. on $(0, \infty) \times \mathbb{N}^2 \times (0, \infty)$ such that J_1 and J_2 can be given by the projections of N_1.

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_1 \xi(s-)} (z_1 - 1) N_1(ds, dz, du), \quad (6)$$

$$\eta(t) = \eta(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_2 \xi(s-)\eta(s-)} (z_2 - 1) N_1(ds, dz, du), \quad (7)$$

where $z = (z_1, z_2) \in \mathbb{N}^2$ and $(\xi(0), \eta(0)) \in \mathbb{N}^2$.

$\xi(\cdot)$ and $\eta(\cdot)$ may involve dependent branching mechanism.

Introduce immigration structure

- Suppose that $N_0(ds, dz)$ is a Poisson r.m. on $(0, \infty) \times \mathbb{N}^2$ independent of N_1. Consider the stochastic equations:

$$\begin{aligned} \xi(t) &= \xi(0) + \int_0^t \int_{\mathbb{N}^2} z_1 N_0(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_1 \xi(s-)} (z_1 - 1) N_1(ds, dz, du), \end{aligned} \tag{8}$$

$$\begin{aligned} \eta(t) &= \eta(0) + \int_0^t \int_{\mathbb{N}^2} z_2 N_0(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_2 \xi(s-) \eta(s-)} (z_2 - 1) N_1(ds, dz, du). \end{aligned}$$

Introduce additional mechanism for the reactant

- Let $r \geq 0$. Suppose that $N_2(ds, dz_2, du)$ is a Poisson r.m. on $(0,\infty) \times \mathbb{N} \times (0,\infty)$ independent of N_0 and N_1.

$$\begin{aligned}\eta(t) &= \eta(0) + \int_0^t \int_{\mathbb{N}^2} z_2 N_0(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_2 \xi(s-)\eta(s-)} (z_2 - 1) N_1(ds, dz, du) \\ &\quad + \int_0^t \int_{\mathbb{N}} \int_0^{r\eta(s-)} (z_2 - 1) N_2(ds, dz_2, du). \qquad (9)\end{aligned}$$

- some reactant particles are not catalyzed by the catalyst.

- We call $(\xi(\cdot), \eta(\cdot))$ defined by (8) and (9) a catalytic DBI-process.

2. Diffusion approximation: a simple case

- Consider a sequence of catalytic DB-processes (without immigration) given by

$$\xi_n(t) = \xi_n(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{n\xi_n(s-)} (z_1 - 1) N_1(ds, dz, du),$$

$$\eta_n(t) = \eta_n(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{\xi_n(s-)\eta_n(s-)} (z_2 - 1) N_1(ds, dz, du),$$

where N_1 is a Poisson r.m. with intensity $ds\mu(dz)du$.

- the offspring distribution μ is defined by the generating function $g(\lambda_1, \lambda_2) = \frac{1}{4}(\lambda_1 + \lambda_2)^2$, which means the dependent branching mechanisms.

- Consider $(\xi_n(t), \eta_n(t))$ in the natural scaling :

$$x_n(t) = \frac{\xi_n(t)}{n} \quad \text{and} \quad y_n(t) = \frac{\eta_n(t)}{n}. \tag{10}$$

Theorem 1 *If $(x_n(0),\ y_n(0))$ converges in distribution to $(x(0), y(0))$, then $(x_n(\cdot), y_n(\cdot))$ converges in distribution on $D([0,\infty), \mathbb{R}^2_+)$ to a process $(x(\cdot), y(\cdot))$, which solves*

$$x(t) = x(0) + \int_0^t \int_0^{x(s)} \sigma_1 W(ds, du), \tag{11}$$

$$y(t) = y(0) + \int_0^t \int_0^{x(s)y(s)} \sigma_2 W(ds, du), \tag{12}$$

where $W(ds, du)$ is a white noise with intensity $dsdu$.

Some remarks

● The diffusion limit can not always represented by (1)-(2). The difficulty comes from dependent branching mechanism.

● a proper representation should be (11) and (12).

EL Karoui-Méléard ('90): vector square-integrable martingales as stochastic integrals of orthogonal martingale measures.

the pathwise uniqueness of solutions of (11)-(12) can be easily proved by the method similar to the Yamada-Watanabe one.

● To get (1)-(2), we have to choose another different scaling to erase the effect of the dependent branching mechanisms.

$$\xi_n(t) = \xi_n(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{n\xi_n(s-)} (z_1 - 1) N_1(ds, dz, du),$$

$$\eta_n(t) = \eta_n(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{n\xi_n(s-)\eta_n(s-)} (z_2 - 1) N_1(ds, dz, du).$$

- $(\frac{\xi_n(\cdot)}{n}, \frac{\eta_n(\cdot)}{n^2}) \Longrightarrow (x(\cdot), y(\cdot))$, which is solved by (1)-(2),

but where $B_1(\cdot)$ and $B_2(\cdot)$ are independent!

Diffusion approximation with jumps

- Consider catalytic DBI-processes in the natural scaling:

$$x_n(t) := \frac{\xi_n(t)}{n} \quad \text{and} \quad y_n(t) := \frac{\eta_n(t)}{n}.$$

- The offspring distribution μ_n is given by the p. g. f. g_n.

For $0 \leq \lambda_1, \lambda_2 \leq n$, set

$$R_n(\lambda_1, \lambda_2) = n\gamma_n\Big[g_n\Big(1 - \frac{\lambda_1}{n}, 1 - \frac{\lambda_2}{n}\Big) - \Big(1 - \frac{\lambda_1}{n}\Big)\Big(1 - \frac{\lambda_2}{n}\Big)\Big].$$

(A) The sequence $\{R_n\}$ is uniformly Lipschitz in (λ_1, λ_2) on each bounded rectangle and $R_n \to R$ as $n \to \infty$;

Theorem 2 *Under (A) and some other technical conditions, $(x_n(\cdot), y_n(\cdot))$ converges in distribution on $D([0,\infty), \mathbb{R}_+^2)$ to a process $(x(\cdot), y(\cdot))$, which defined by*

$$
\begin{aligned}
x(t) \;=\; & x(0) + \int_0^t (b_1 + \beta_{11} x(s)) ds + \int_0^t \int_0^{x(s-)} \sigma_{11} W_1(ds, du) \\
& + \int_0^t \int_0^{x(s-)} \sigma_{12} W_2(ds, du) + \int_0^t \int_{\mathbb{R}_+^2} z_1 N_0(ds, dz) \\
& + \int_0^t \int_{\mathbb{R}_+^2} \int_0^{x(s-)} z_1 \tilde{N}_1(ds, dz, du)
\end{aligned}
$$

$$
\begin{aligned}
&y(t) \\
&= y(0) + \int_0^t (b_2 + \beta_{21} x(s) y(s) + \beta_{22} y(s)) ds \\
&+ \int_0^t \int_0^{y(s-)} \sigma_0 W_0(ds, du) + \int_0^t \int_0^{x(s-)y(s-)} \sigma_{21} W_1(ds, du) \\
&+ \int_0^t \int_0^{x(s-)y(s-)} \sigma_{22} W_2(ds, du) + \int_0^t \int_{\mathbb{R}_+^2} z_2 N_0(ds, dz) \\
&+ \int_0^t \int_{\mathbb{R}_+^2} \int_0^{x(s-)y(s-)} z_2 \tilde{N}_1(ds, dz, du),
\end{aligned}
$$

where $\tilde{N}_1(ds, dz, du) = N_1(ds, dz, du) - ds\mu(dz)du$.

- $(x(\cdot), y(\cdot))$ can be naturally regarded as a general catalytic CBI-process.

2. Fluctuation limits and affine Markov processes

- Let $D = \mathbb{R}_+ \times \mathbb{R}$ and $U = \mathbb{C}_- \times (i\mathbb{R})$. A Markov semigroup $(P_t)_{t\geq 0}$ on D is said to be affine if

$$\int_D \exp\{\langle u, \xi\rangle\} P_t(x, d\xi) = \exp\{\langle x, \psi(t,u)\rangle + \phi(t,u)\} \quad (13)$$

for all $u \in U$.

- Duffie-Filipović-Schachermayer ('03) showed that affine processes have a wide range of applications in finance.

● The affine diffusion $(X(\cdot), Y(\cdot))$ is described by the following stochastic equations

$$\begin{aligned} X(t) &= X(0) + \int_0^t (b_1 + \beta_{11} X(t))\, dt + \int_0^t \sigma_{11} \sqrt{X(t)}\, dB_1(t) \\ &\quad + \int_0^t \sigma_{12} \sqrt{X(t)}\, dB_2(t), \\ Y(t) &= Y(0) + \int_0^t (b_2 + \beta_{21} X(t) + \beta_{22} Y(t))\, dt + \int_0^t \sqrt{a}\, dB_0(t) \\ &\quad + \int_0^t \sigma_{21} \sqrt{X(t)}\, dB_1(t) + \int_0^t \sigma_{22} \sqrt{X(t)}\, dB_2(t), \end{aligned}$$

where $a, b_1 \geq 0$ and $b_2, \beta_{11}, \beta_{21}, (\sigma_{ij})$, and where $B(\cdot) = \big(B_0(\cdot), B_1(\cdot), B_2(\cdot)\big)$ is a three-dimensional Brownian motion. See Dawson and Li ('06) for a general case.

The high density fluctuation limit

- Consider another sequence of catalytic DBI-processes by

$$\xi_n(t) = \xi_n(0) + \int_0^t \int_{\mathbb{N}^2} z_1 N_{n,0}(ds, dz) + \int_0^t \int_0^{\gamma_n \xi_n(s-)} \int_{\mathbb{N}^2} (z_1 - 1) N_{n,1}(ds, du, dz),$$

$$\eta_n(t) = \eta_n(0) + \int_0^t \int_{\mathbb{N}^2} z_2 N_{n,0}(ds, dz) + \int_0^t \int_0^{\frac{\gamma_n}{n^2} \xi_n(s-) \eta_n(s-)} \int_{\mathbb{N}^2} (z_2 - 1) N_{n,1}(ds, du, dz) + \int_0^t \int_{\mathbb{N}} \int_0^{\theta_n \eta_n(s-)} (z_2 - 1) N_{n,2}(ds, dz_2, du).$$

- Let $x_n(t) = \frac{\xi_n(t)}{n}$ and $y_n(t) = \frac{\eta_n(t) - n^2}{n}$.

Theorem 3 *Under suitable conditions, $(x_n(t), y_n(t))$ converges as $n \to \infty$ to an affine process with non-negative jumps $(x(t), y(t))$ with state space $\mathbb{R}^+ \times \mathbb{R}$.*

Conversely, any affine process with non-negative jumps arise in a fluctuation limit for some sequence of catalytic DBI-processes.

- Idea: The main feature of catalytic DBI-processes is a non-linearity. Re-scale this non-linearity to affine linearity.

Some references

1. Dawson; Fleischmann: *J. Theoret. Probab.* **10** ('97).

2. Dawson, D.A. and Li, Z.H.: *Ann. Probab.* **34** ('06).

3. Duffie; Filipović; Schachermayer: *Ann. Appl. Probab.* **13** ('03).

4. El Karoui, N.and Méléard, S.: Martingale measures and stochastic calculus. *Probab. Theory Related Fields.* **84** (1990), 83–101.

5. Kawazu, K. and Watanabe, S.: Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16** (1971), 36-54.

6. Li, Z.H.: Ornstein-Uhlenbeck type processes and branching processes with immigration. *J. Appl. Probab.* **37** (2000), 627-634.

Thanks!