Regularity of Solutions to Differential Equations with non-Lipschitz Coefficients

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1 Outline

- **Background.**
- **Our results.**
- **Sketch of Proofs.**

2 Background

Equations

Let $A_0, A_1, \cdots, A_N$ be vector fields on $\mathbb{R}^d$ and $w_t = (w_t^1, \cdots, w_t^N)$ a standard Brownian motion on $\mathbb{R}^N$. Consider the ODE

$$dX_t = A_0(X_t)dt, \quad X_0 = x \tag{1}$$

and the Itô SDE

$$dY_t = \sum_{k=1}^{N} A_k(Y_t)dw_t^k + A_0(Y_t)dt, \quad Y_0 = x. \tag{2}$$

The classical theory of ODE (resp. SDE) requires that A_0 (resp. $A_0, \cdots, A_N$) satisfies global Lipschitz condition. Recently the conditions have been weakened in two directions: divergence-type conditions and non-Lipschitz conditions.

First direction: Divergence-type conditions

λ: Lebesgue measure on $\mathbb{R}^d$.
σ: Standard Gaussian measure on $\mathbb{R}^d$.

Diperna and Lions (1989): If $A_0 \in W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ and $\text{div}(A_0)$ is bounded, then ODE (1) defines a flow of measurable maps, which leaves λ quasi-invariant, i.e. the push-forward $\lambda_t = \lambda \circ X_t^{-1} \ll \lambda$.

Cipriano and Cruzeiro (2005): Similar result holds if $A_0 \in W^{1,1}_{loc}(\sigma)$ and there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^d} e^{\delta|\text{div}_\sigma(A_0)|} d\sigma < \infty, \tag{3}$$

where $\text{div}_\sigma(A_0) = \text{div}(A_0) + x \cdot A_0$ is the divergence with respect to σ.

Note that although the second result allows $\text{div}_\sigma(A_0)$ to be unbounded, it does not cover that of the first completely. For example, let $A_0 = \big((x-y)^2, (x-y)^2\big)$, then $\text{div}(A_0) = 0$ but (3) is not satisfied.

Second direction: Non-Lipschitz Condition

For some $\beta \in (0,1]$, there exist C, $c_0 > 0$ such that for all $x, y\mathbb{R}^d$ with $|x-y| \le c_0$,

$$\sum_{k=1}^{N} |A_k(x) - A_k(y)|^2 \le C|x-y|^2 \left(\log \frac{1}{|x-y|} \right)^{\beta}, \qquad (i)$$

$$|A_0(x) - A_0(y)| \le C|x-y| \left(\log \frac{1}{|x-y|} \right)^{\beta}. \qquad (ii)$$

c_0 is small enough so that $s \to s\left(\log \frac{1}{s}\right)^{\beta}$ is increasing and concave on $(0, c_0)$.

We always assume the linear growth condition

$$\sum_{k=1}^{N} |A_k(x)|^2 \le C_0(1+|x|^2), \quad |A_0(x)| \le C_0(1+|x|), \qquad (iii)$$

C_0 is a constant.

Known results and question

Shizan Fang and Tusheng Zhang (2005): Under the conditions (i), (ii) with $\beta = 1$ and (iii), Itô equation (2) has a unique strong solution which is continuous on $[0, \infty) \times \mathbb{R}^d$.

Xicheng Zhang (2005): Under slightly stronger conditions: (i), (ii) hold with $\beta = 1$ and with $\log \frac{1}{|x-y|}$ being replaced by $1 \vee \log \frac{1}{|x-y|}$, then Y_t is a stochastic flow of homeomorphisms on $\mathbb{R}^d$.

Our question: What are the regular properties of X_t (or Y_t), i.e. to which extent X_t (or Y_t) is continuously dependent on the initial value x.

3 Our results

Proposition 1. (ODE) (a) Assume (ii) with $\beta \in (0,1)$ and (iii). Then for all $t > 0$, $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is α-Hölder continuous for any $\alpha \in (0,1)$.

(b) Assume (ii) with $\beta = 1$ and (iii), then X_t is Hölder continuous of order e^{-Ct}.

Proposition 2. (SDE) Assume A_i, $0 \le i \le N$ satisfy (i), (ii) with $\beta \in (0,1)$ and (iii). Then Y_t has a version $\tilde{Y}_t$ such that a.s., for all $t \ge 0$ and $\alpha \in (0,1)$, $\tilde{Y}_t$ is α-Hölder continuous on any ball $B(R)$.

Comparison of Results

ODE (1):

- A_0 is globally Lipschitz continuous, then so is X_t;
- A_0 satisfies (ii) with $\beta < 1$ and (iii), then $X_t \in C^\alpha$ for any $\alpha \in (0,1)$;
- A_0 satisfies (ii) with $\beta = 1$ and (iii), then the order of Hölder continuity of X_t decreases exponentially fast along the time.

SDE (2):

- If $A_0, \cdots, A_N$ satisfy global Lipschitz condition, using Kolmogorov's modification theorem, we can only prove that a.s. $\forall\, t > 0$, Y_t is α-Hölder continuous for any $\alpha \in (0,1)$.
- If (i), (ii) hold with $\beta < 1$ and (iii), then we also have a.s. $\forall\, t > 0$, Y_t is α-Hölder continuous for any $\alpha \in (0,1)$.

Hence in the situation of SDE, we may say

$$\beta < 1,\ (i) + (ii) + (iii) \approx \text{global Lipschitz condition.}$$

Corollary

$\dim_H(E)$: the Hausdorff dimension of $E \subset \mathbb{R}^d$.

For ODE (1), the inverse of X_t satisfies the following equation

$$dX_t^{-1} = -A_0\big(X_t^{-1}\big)dt, \quad X_0^{-1} = x.$$

Since $-A_0$ has exactly the same properties as those of A_0, the results in Proposition 1 also hold for the map $X_t^{-1} : \mathbb{R}^d \to \mathbb{R}^d$. Therefore we have

Corollary. (a) Under the conditions of Proposition 1 (a), for any $E \subset \mathbb{R}^d$ and $t > 0$,

$$\dim_H(X_t(E)) = \dim_H(E).$$

(b) Under the conditions of Proposition 2, for any $E \subset \mathbb{R}^d$, a.s. for all $t > 0$,

$$\dim_H(Y_t(E, w)) \le \dim_H(E).$$

An estimate of div(A_0)

Suppose div(A_0) exists in the sense of distribution, i.e. for every $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \langle A_0, \nabla\psi\rangle d\lambda = -\int_{\mathbb{R}^d} \mathrm{div}(A_0)\,\psi\, d\lambda. \tag{4}$$

This is weaker than the requirement that $A_0 \in W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. We have a simple estimate:

Proposition 3. Assume (ii) and div$(A_0) \in L^1_{loc}(\mathbb{R}^d)$ exists in the sense of (4), then $\exists\, C_d > 0$ such that $\forall$ ball $B(x,r)$, we have

$$\left|\int_{B(x,r)} \mathrm{div}(A_0) d\lambda\right| \le C_d \lambda(B(x,r)) \left(\log \frac{1}{\lambda(B(x,r))}\right)^{\beta}. \tag{5}$$

Remark. If $B(x,r)$ is replaced by d-dimensional small cubes, the estimate (5) also holds with another constant C_d.

Main Result

Theorem. Assume (ii) with $\beta = 1$, (iii) and $\mathrm{div}(A_0) \in L^1_{loc}(\mathbb{R}^d)$ exists in the distribution sense.

(i) If $\mathrm{div}(A_0)$ is bounded, then the Lebesgue measure is quasi-invariant under the flow X_t.

(ii) If there exist $\gamma \in (0,1)$ and $C > 0$ such that for any small connected open subset $O \subset \mathbb{R}^d$, we have

$$\left| \int_O \mathrm{div}(A_0)(x) d\lambda \right| \le C\lambda(O) \left(\log \frac{1}{\lambda(O)} \right)^\gamma, \tag{6}$$

then for any subset $E \subset \mathbb{R}^d$ with $\dim_H(E) < d$, it holds that $\lambda_t(E) = 0$ for all $t \ge 0$.

Remark. We do not require that $A_0 \in W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. In the proof of (ii), the "connected open set" is $X_t^{-1}(B(x,r))$. Since X_t^{-1} is a homeomorphism, when the radius r is small enough, $X_t^{-1}(B(x,r))$ are very close to standard balls.

4 Sketch of the proof of Theorem (i).

If $A_0 \in C_b^1$ and $\theta_0 \in C^1$, then the solution $\theta(t,x)$ to the transport equation

$$\partial_t \theta + A_0 \cdot \nabla \theta = 0, \quad \theta(0,\cdot) = \theta_0$$

can be expressed by $\theta(t,x) = \theta_0\big(X_t^{-1}(x)\big)$, where $X_t^{-1}(x)$ is the inverse flow of the solution to the following ODE

$$dX_t = A_0(X_t)dt, \quad X_0 = x.$$

Now under the non-Lipschitz condition

$$|A_0(x) - A_0(y)| \leq C|x-y| \log \frac{1}{|x-y|},$$

$\forall\, t > 0,\ X_t : x \to X_t(x)$ is a homeomorphism of $\mathbb{R}^d$. Approximating A_0 by smooth vector fields, we can prove $\theta(t,x) = \theta_0\big(X_t^{-1}(x)\big)$ still solves the transport equation, but in the distribution sense, i.e. for any $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \theta(t,x)\psi(x)dx = \int_{\mathbb{R}^d} \theta_0(x)\psi(x)dx - \int_0^t \int_{\mathbb{R}^d} \theta(s,x)\mathrm{div}(\psi A_0)(x)dx.$$

Fix $T > 0$ and $\theta_0 \in C_c(\mathbb{R}^d)$. Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi \equiv 1$ on the union $\bigcup_{0 \le t \le T} X_t^{-1}(\mathrm{supp}(\theta_0))$. Then $\forall\, t \le T$,

$$\int_{\mathbb{R}^d} \theta(t,x)dx = \int_{\mathbb{R}^d} \theta_0(x)dx - \int_0^t \int_{\mathbb{R}^d} \theta(s,x)\mathrm{div}(A_0)(x)dxds$$

$$\Longleftrightarrow \int_{\mathbb{R}^d} \theta_0\, d\lambda_t = \int_{\mathbb{R}^d} \theta_0\, d\lambda - \int_0^t \int_{\mathbb{R}^d} \theta_0\, \mathrm{div}(A_0)\big(X_s^{-1}(x)\big)d\lambda_s ds.$$

Now let U be a "regular" subset of $\mathbb{R}^d$. Approximating $\mathbf{1}_U$ by $\theta_0 \in C_c(\mathbb{R}^d)$, we get

$$\lambda_t(U) = \lambda(U) - \int_0^t \int_U \mathrm{div}(A_0)\big(X_s^{-1}(x)\big)d\lambda_s(x)ds.$$

Therefore

$$\lambda_t(U) \le \lambda(U) + \|\mathrm{div}(A_0)\|_\infty \int_0^t \lambda_s(U)ds.$$

And Gronwall's inequality gives us

$$\lambda_t(U) \le \lambda(U)e^{\|\,\mathrm{div}(A_0)\|t}.$$

Thank you very much!