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A branching random walk on $\mathbb{R}$ in random environment

LIU Quansheng

South-Brittany Univ.

(Univ. de Bretagne-Sud, France)

and

Changsha Univ. of Sci. and Tech., China

1. Introduction and description of the model

Branching random walk (BRW) on $\mathbb{R}$ (or $\mathbb{Z}$) in random environment:

a system of particles on $\mathbb{R}$ (or $\mathbb{Z}$), where each particle reproduces new ones moving on $\mathbb{R}$ (or $\mathbb{Z}$); the offspring distribution and the moving law of a particle depend on an environment in time and/or in locations. Different models have been studied according to the dependence on the environment in time and/or in locations.

Our model: Branching random walk on $\mathbb{R}$ with random environment in time:

the offspring distribution of a particle of generation n, and the distributions of the displacements of their children depend on an environment ξ_n, indexed by the time n, supposed to be stationary and ergodic.

The difference with the classical BRW:
the distributions are realizations of a stochastic process, rather than fixed ones.

Description of the model:

Random environment (in time):

modeled as a sequence of stationary and ergodic random variables with values in some measurable space (Θ, F).

Each value of ξ_n corresponds to point distribution $\mu_n = \mu(\xi_n)$ on $\mathbb{R}$.

At generation 0, there is one initial particle located at $S_\emptyset = 0 \in \mathbb{R}$.

Each particle u of gen. n located at S_u is replaced by N_u new particles ui of gen. $n+1$, located at

$$S_{ui} = S_u + L_{ui} \quad (1 \leq i \leq N_u);$$

the point process formulated by the the number of offspring and their displacements, $(N_u; L_{u1}, ..., L_{uN_u})$, is of distribution $\mu_n = \mu(\xi_n)$ (given the environment ξ).

All particles behave independently: given ξ, $(N_u; L_{u1}, ..., L_{uN_u})$ are independent of each other.

Closely related topics: Classical Branching Random Walks: Biggins (1977, 1978), R. Lyons (1997), ...

Branching processes: Harris (1963), Smith, W.L. and Wilkinson, W. (1969), Athreya and Karlin (1971), ...

Mandelbrot's multiplicative cascades: J. P. Kahane and J. Peyriére (1976), Y. Guivarc'h (1990), Q. Liu (2000), ... ;

Infinite particle systems: Durrett and Liggett (1983), ...

Quicksort algorithms: Rösler (1993), ...

Random fractals: Mauldin and Williams (1986), Falconer (1986), Q. Liu (1993) ...

Weighted branching processes: D. Kuhlbusch (2004), ...

Other topics: see Q. Liu (1998), and D. J. ALDOUS AND A. BANDYOPADHYAY (2005).

Quenched and annealed laws:

Let (Γ, P_ξ) be the probability space under which the process is defined when the environment ξ is fixed. As usual, P_ξ is called *quenched law*.

The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive g we have

$$\int g dP = \int \int g(\xi, y) dP_\xi(y) d\tau(\xi),$$

where τ is the law of the environment ξ. P is called *annealed law*. P_ξ may be considered to be the conditional probability of P given ξ.

Related models: random walks on $\mathbb{Z}$ in random environment:

(1) Many authors considered the case where the offspring distribution of a particle situated at $z \in \mathbb{Z}$ depends on a random environment indexed by z, while the moving mechanism is controlled by a fixed deterministic law. See for example Greven and der Hollander (1992), Baillon, Clement, Greven and den Hollander (1992), Fleischmann and Greven (1992), and Révśz (1998).

(2) Devulder (2005) considered the case where the offspring distribution is fixed, while the moving law of a particle situated at z depends on a random environment indexed by z.

(3) Comets, Menshikov and Popop (1998), and Machado and Popov (2000) considered a branching random walk on $\mathbb{N}$ rather then $\mathbb{Z}$, where both offspring distributions and the moving laws depend on a random environment (ω_z) indexed by locations.

(4) Xu Li, Yingqiu Li and Quansheng Liu (2007) considered a branching random walk on $\mathbb{Z}$ where the offspring distribution of a particle of generation n and located at $z \in \mathbb{Z}$ depend on the environment in time ξ_n, while the moving laws of its children depend on an environment in locations, ω_z.

Main problems that we study:

Recall that S_u denotes the position of a particle u. For $n \in \mathbb{N}$, let T_n be the set of particles of generation n, and let

$$Z_n = \sum_{u \in T_n} \delta_{S_u}$$

be the counting measure of particles of gen. n, so that for $A \subset \mathbb{R}$,

$$Z_n(A) = \text{number of particles of gen. n located in} A.$$

Let

$$L_n = \min_{u \in T_n} S_u \quad (resp. R_n = \max_{u \in T_n} S_u)$$

be the position of leftmost (resp. rightmost) particles of gen. n.

We consider the case where the corresponding branching process in random environment, $(Z_n(\mathbb{R}))$, is supercritical, and we want to know asymptotic properties of

$$Z_n(A), \quad L_n \quad \text{and } R_n$$

2. The mean of $Z_n(nA)$

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, let

$$m_n(t) = E_\xi \sum_{i=1}^{N_u} e^{tL_{ui}}, \tag{1}$$

where u is a sequence of length n; by convention, $\emptyset$ is of length 0, and $\emptyset i = i$. Notice that the expectation does not depend on the choice of u. Then

$$m_0(t) = E_\xi \sum_{i=1}^{N} e^{tL_i}, \quad m_0(0) = E_\xi N.$$

Throughout the paper we will always assume that

$$E \log m_0(0) > 0 \text{ and } E|\log m_0(t)| < \infty \tag{2}$$

for all $t \in \mathbb{R}$, so that the corresponding BPRE $Z_n(\mathbb{R})$ is supercritical:

$$Z_n(\mathbb{R}) \to \infty$$

with positive probability. For simplicity, assume also that a.s.

$$N \geq 1.$$

Let

$$\Lambda(t) := E \log m_0(t),$$

and let

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{xt - \Lambda(t)\} \tag{3}$$

be its Legendre transform. Then Λ is differentiable on $\mathbb{R}$,

$$\Lambda^*(x) = \begin{cases} t\Lambda'(t) - \Lambda(t) & if \quad x = \Lambda'(t) \text{ for some } t \in \mathbb{R}, \\ +\infty & if \quad x > \Lambda'(+\infty) \text{ or } x < \Lambda'(-\infty), \end{cases} \tag{4}$$

and

$$\min_x \Lambda^*(x) = \Lambda^*(\Lambda'(0)) = -\Lambda(t) = -E \log m_0(0) < 0.$$

Theorem 1 (Large Deviation Principle for quenched mean). For almost every ξ, the sequence of finite measures $A \mapsto E_\xi Z_n(nA)$ satisfies a principle of large deviation with rate function Λ^*: for each measurable subset A of $\mathbb{R}$,

$$
\begin{aligned}
-\inf_{x \in A^o} \Lambda^*(x) &\leq \liminf_{n \to \infty} \frac{1}{n} \log E_\xi Z_n(nA) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \log E_\xi Z_n(nA) \leq -\inf_{x \in \bar{A}} \Lambda^*(x),
\end{aligned}
$$

where A^o denotes the interior of A, and $\bar{A}$ its closure.

Proof. Let $q_n(A) = E_\xi Z_n(A)$. Show that

$$\tilde{q_n}(t) := \int e^{tx} dq_n(x) = E_\xi \sum_{u \in \mathbb{T}_n} e^{tS_u} = m_0(t) ... m_{n-1}(t), \tag{5}$$

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{q_n}(t) = \Lambda(t) := E \log m_0(t) \text{ a.s.} \tag{6}$$

and apply Gärtner-Ellis' theorem to the sequence of normalized probability measures $q_n(nA)/q_n(\mathbb{R})$.

Let $\Lambda_a(t) = \log Em_0(t)$ and Λ_a^* be its Legendre transform. Similarly, we have:

Theorem 2 (Large Deviation Principle for annealed mean). Assume that ξ_n are iid. Then the sequence of finite measures $A \mapsto EZ_n(nA)$ satisfies a principle of large deviation with rate function Λ_a^*: for each measurable subset A of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in A^o} \Lambda_a^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log EZ_n(nA) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log EZ_n(nA) \leq -\inf_{x\in \bar{A}} \Lambda_a^*(x), \end{aligned}$$

where A^o denotes the interior of A, and $\bar{A}$ its closure.

Remark:

$$\boldsymbol{\Lambda_a(t) \geq \Lambda(t)} \quad \text{and} \quad \boldsymbol{\Lambda_a^*(x) \leq \Lambda^*(x),}$$

so that the growth rate of the annealed mean $\boldsymbol{EZ_n[an,\infty)}$ is greater than that of the quenched mean $\boldsymbol{E_\xi Z_n[an,\infty)}$.

3. Convergence of the free energy

Let

$$\tilde{Z}_n(t) := \int e^{tx} dZ_n(x) = \sum_{u \in T_n} e^{tS_u}$$

be the Laplace transform of Z_n, also called the partition function. We are interested to the convergence of the free energy $\frac{\log \tilde{Z}_n(t)}{n}$, and the asymptotic properties of $Z_n(nA)$.

Definition of t_- and t_+

Let

$$\rho(t) = t\Lambda'(t) - \lambda(t), \qquad t \in \mathbb{R}.$$

Then

$$\rho'(t) = t\Lambda''(t),$$

$\rho(t)$ decreases on $\mathbb{R}_-$, increases on $\mathbb{R}_+$, and attains its minimum at 0:

$$\min_t \rho(t) = \rho(0) = -\lambda(0) < 0.$$

Let

$$t_- = \inf\{t \in \mathbb{R} : t\Lambda'(t) - \lambda(t) \leq 0\},$$

$$t_+ = \sup\{t \in \mathbb{R} : t\Lambda'(t) - \lambda(t) \leq 0\}.$$

Then $-\infty \leq t_- < 0 < t_+ \leq \infty$, t_- and t_+ are two solutions of $t\Lambda'(t) - \lambda(t) = 0$ if they are finite.

Theorem 3 (Convergence of the free energy). A.s.

$$\lim_{n \to \infty} \frac{\log \tilde{Z}_n(t)}{n} = \tilde{\Lambda}(t) := \begin{cases} \Lambda(t) & if \quad t \in (t_-, t_+) \\ t\Lambda'(t_+) & if \quad t \geq t_+ \\ t\Lambda'(t_-) & if \quad t \leq t_- \end{cases}$$

Deterministic environment case: B. Chauvin and A. Rouault (1997), J. Franchi (1993).

4. Almost sure asymptotic properties of $Z_n(nA)$:

Let $\tilde{\Lambda}^*(x)$ be the Legendre transform of $\tilde{\Lambda}(t)$. By Theorem 3 and Gärtner- Ellis' theorem, we obtain:

Theorem 4 (LDP for $Z_n(nA)$). A.s. the sequence of finite measures $A \mapsto Z_n(nA)$ satisfies a principle of large deviation with rate function $\tilde{\Lambda}^*$: for each measurable subset A of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x \in A^o} \tilde{\Lambda}^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log Z_n(nA) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log Z_n(nA) \leq -\inf_{x \in \bar{A}} \tilde{\Lambda}^*(x), \end{aligned}$$

where A^o denotes the interior of A, and $\bar{A}$ its closure.

Corollary A.s.

$$\lim_n \frac{1}{n} \log Z_n[nx, \infty) = -\Lambda^*(x) > 0 \text{ if } x \in (\Lambda'(0), \Lambda'(t_+)),$$

$$\lim_n \frac{1}{n} \log Z_n(-\infty, nx] = -\Lambda^*(x) > 0 \text{ if } x \in (\Lambda'(t_-), \Lambda'(0)).$$

Remark.

$$x \in (\Lambda'(0), \Lambda'(t_+) \text{ iff } x > \Lambda'(0) \text{ and } \Lambda^*(x) < 0.$$

$$x \in (\Lambda'(t_-), \Lambda'(0) \text{ iff } x < \Lambda'(0) \text{ and } \Lambda^*(x) < 0.$$

For deterministic branching random walk: see Biggins (1977).

5. Positions of the rightmost and leftmost particles of gen. n

Theorem 5 (Asymptotic properties of L_n and R_n) It is a.s. that

$$\lim_n \frac{L_n}{n} = \Lambda'(t_-),$$

$$\lim_n \frac{R_n}{n} = \Lambda'(t_+).$$

For deterministic branching random walk: see Biggins (1977).

6 Proof of theorems 3 and 5

Observation:

$$\boldsymbol{W_n(t) := \frac{\tilde{Z}_n(t)}{E_\xi \tilde{Z}_n(t)} = \frac{\sum_{u \in T_n} e^{tS_u}}{m_0(t) \ldots m_{n-1}(t)}}$$

is a martingale, therefore converges a.s. to a r.v. $\boldsymbol{W(t) \in [0, \infty)}$.

Remark: In the constant environment case, this martingale has been studied by J. P. Kahane - J. Peyrière (1976), J. Biggins (1977), Durrett-Liggett (1983), Y. Guivarc'h (1990), R. Lyons (1997) and Q. Liu (1997, 1998, 2000, 2001), etc. in different contexts.

Lemma 1 If $t \in (t_-, t_+)$ and $EW_1(t)\log^+ W_1(t) < \infty$, then

$$W(t) > 0 \quad \text{a.}s.$$

If $t \le t_-$ or $t \ge t_+$), then

$$W(t) = 0 \quad \text{a.}s.$$

Proof. Use a result by Dirk Kuhlbusch (2004) on weighted branching processes in random environment.

Lemma 2 A.s.

$$\limsup_n \frac{R_n}{n} \leq \Lambda'(t_+)$$

Proof. For $a > \Lambda'(t_+)$, use asymptotic properties of $E_\xi Z_n[an,\infty)$ to show that

$$\sum_n P_\xi(Z_n[an,\infty) \geq 1) < \infty,$$

and apply Borel-Cantelli's lemma to conclude that P_ξ a.s.

$$Z_n[an,\infty) = 0 \quad \text{for n large enough.}$$

Lemma 3 If $t \geq t_+$, then

$$\limsup_n \frac{\log \tilde{Z}_n(t)}{n} \leq t\Lambda'(t_+)$$

Thank you!

Quansheng.Liu@univ-ubs.fr