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Theory of Anticipating Local Times

(非适应局部时理论)

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1 Brownian Motions

Brownian Motions

We assume as given an underlying complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual hypotheses, i.e., $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right continuous.

Definition 1: Brownian Motions

The Gaussian stochastic processes $\{W(t)\}$ satisfying the following three properties

(i) $W(0) = 0$,

(ii) $\mathbf{E}(W(t)) = 0$ for all $t \geq 0$,

(iii) $\mathbf{E}(W(t)W(s)) = \frac{1}{2}[|t| - |t-s| + |s|]$ for all $s, t \geq 0$

are called the standard Brownian Motions. It is well known that the Brownian Motions have the following properties

$$\mathbf{E}(W(t) - W(s))^2 = (t - s), \tag{1}$$

$$\lim_{|\triangle_n|\to 0} \sum_{t_i\in\triangle_n} (W(t_{i+1}) - W(t_i))^2 = t - s, \tag{2}$$

where $(\triangle_n)$ is a sequence of subdivisions of $[a, b]$ such that $|\triangle_n| \longrightarrow 0$.

Three Directions of Stochastic Integrals $\int_0^t f dY(s)$

1. The case when Y is a Gaussian process(martingale)

Itô's calculus (Kiyosi Itô; others);

Stochastic integral w.r.t. martingale (P.A. Meyer; others);

Malliavin calculus (Paul, Malliavin; J.M. Bismut; D.W.Stroock; others);

Stochastic integral w.r.t.fractional Brownian motions (David Nualart; others).

2. The case when Y is a Dirichlet process

Theory of Dirichlet form(Masatoshi Fukushima; Zhi-Ming Ma; others).

3. The case when Y has paths with p-variation

Theory of stochastic integral w.r.t. the rough path (T.J. Lyons; others).

Itô's Formula

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds, \qquad f \in C^2(R).$$

Reflecting Brownian Motions(Diffusion Processes)

$$|W_t| = x_0 + W_t + A_t.$$

Fractional Sobolev spaces $\mathbb{D}^{\alpha,p}$ and Besov spaces $\mathcal{B}^{\alpha}_{p,q}$

Denote Malliavin derivative by $\mathcal{D}$ on the Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. The fractional Sobolev spaces $\mathbb{D}^{\alpha,p}$ can be defined as intermediate spaces between $\mathbb{D}^{0,p}$(i.e., $L^p(\Omega, \mathbf{P})$) and $\mathbb{D}^{1,p}$.

Let $f : [0,1] \longrightarrow \Re$ be a measurable function, $\omega_p(f,t) = \sup_{0\leq h\leq t} \|(f(\cdot + h) - f(\cdot))I_{[0,1-h]}(\cdot)\|_p$ is its modulus $\omega_p(f,t)$ of smoothness. Define

$$\|f\|_{\alpha,p,q} = \begin{cases} \|f\|_p + \left(\int_0^1 (\frac{1}{t^\alpha}\omega_p(f,t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < +\infty, \\ \|f\|_p + \sup\limits_{0\leq t\leq 1} \frac{\omega_p(f,t)}{t^\alpha} & \text{if } q = +\infty. \end{cases} \tag{3}$$

The Besov spaces $\mathcal{B}^{\alpha}_{p,q} = \{f : \|f\|_{\alpha,p,q} < +\infty\}$ and are Banach spaces. $\mathcal{B}^{\alpha,0}_{p,q} = \{f : f \in \mathcal{B}^{\alpha}_{p,q}$ and $\omega_p(f,t) = O(t^\alpha)(t \downarrow 0)\}$ are their closed subspaces.

2 Theory of Adapted Local Times

To my knowledge, there are more than 1000 papers studying Local Times(see Special Invited paper, Ann.Prob.1980,Vol.8,1-67, and etc). Here we summarize three main parts of the theory as follows.

Part one: Existence of Local Times $L(t, x, \omega)$

Proposition 1. If $X_t = \int_0^t u_s dW(s)$ is the Ito integral process (continuous semimartingales). Then there exists an increasing continuous process L^x satisfies the following

(Occupation times formula):

$$\int_0^t \Phi(X_s) d < X, X >_s = \int_{-\infty}^{\infty} \Phi(a) L_t^a da$$

for every t and every positive Borel function Φ, or it satisfies the following (Tanaka formula):

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_s > a\}} dX_s + \frac{1}{2} L_t^a$$

for every t. The process L^a above is called the Local time of X in a.
See Ann.Prob.1980,Vol.8,1-67; D. Revuz; M. Yor ; Continuous Martinglaes and Brownian motion. Chapter VI, Springer-Verlag, ISBN3-540-52167-4.

Part two: Path Properties of Local Times $L(t, x, \omega)$

I. *Fractional Smooth Properties on Spatial Variable* x *for* $L(t, x, \omega)$ *and Time Parameter* t *for Brownian Motion* $W(t)$

Proposition 2.

If u satisfies $\int_0^1 \mathbf{E}|u_s|^{2p} ds < +\infty$ and $u_t(\omega) \neq 0$ almost surely with respect to (t, ω), then

(1) The path $x \longrightarrow L(t, x, \omega)$ ($t \longrightarrow W(t)$) almost surely belongs to the Besov space $\mathcal{B}^{\alpha}_{p,q}$ for $\alpha < \frac{1}{2}, p, q \in [1, \infty]$.

(2) The path $x \longrightarrow L(t, x, \omega)$ ($t \longrightarrow W(t)$) almost surely does not belong to the Besov space $\mathcal{B}^{\frac{1}{2},0}_{p,q}$ for $p \in [1, \infty]$ and $q \in [1, \infty)$.

(3)The path $x \longrightarrow L(t, x, \omega)$ ($t \longrightarrow W(t)$) almost surely does not belong to the Besov space $\mathcal{B}^{\alpha}_{p,q}$ for $\alpha > \frac{1}{2}, p, q \in [1, \infty]$.

See C.R.Acad.Sci.Math.316(1993) 843-848, Ann.Prob.1980,Vol.8,1-67; Bull.Sci.Math.123(1999)643-663;C.R.Acad. Sci.Math.330(2000),719-724.

II. *Pathwise Fractional Smooth Properties on* $L(t,x,\omega)$ *w.r.t path* ω

Proposition 3.

If u satisfies $u \in \bigcap_{1<p<+\infty} \mathbb{D}^{1,p}$ and $\int_0^1 \mathbf{E}||u_s||_{p,1}^p ds < +\infty$ for $p > 1$, then for every $(t,x) \in [0,1] \times \Re$ the local time $\omega \mapsto L(t,x,\omega)$ belongs to the fractional Sobolev spaces $\mathbb{D}^{\alpha,p}$ for $\alpha < \frac{1}{2}$ and $p > 1$. The result is optimal if $u = 1$, i.e, the Brownian motion Local time $L(t,x,\omega)$ is not in $\mathbb{D}^{\frac{1}{2},p}$ for $p > 1$.

See PTRF. 95(2) (1993)175-189. Ann.Inst.H.Poincare (2002), and references therein.

III. *Variational Properties on* $L(t,x,\omega)$ *w.r.t. Spatial Variable* x

Proposition 4.

If u satisfies $\int_0^1 \mathbf{E}|u_s|^p ds < +\infty$ for $p \geq 1$ and $(\triangle_n)$ is a sequence of subdivisions of $[a,b]$ such that $|\triangle_n| \longrightarrow 0$, then

$$\lim_{n\to\infty} \sum_{\triangle_n} (L(t,x_{i+1},\omega) - L(t,x_i,\omega))^2 = 4\int_a^b L(t,x,\omega)dx$$

in probability.

See Ann. Prob. 1992, vol.20, 1685-1713. D. Revuz; M. Yor, Continuous martingales and Brownian motion, and references therein.

Part Three: Reflected Stochastic Differential Equations (Reflecting Brownian motions)

Proposition 5.

Assume that the function σ, its derivatives σ' are Lipschitz continuous on $[0,\infty)$. Then there exists a pair $(X_t(x), L_t^x, t \in [0,1])$ satisfies the following

$$X_t(x) = x + \int_0^t \sigma(X_s(x)) \circ dB_s + L_t^x, \; \forall\, t \in [0,1] \tag{4}$$

(i)$X_0(x) = x$, $X_t(x) \geq 0$ for $t \in [0,1]$,
(ii) $X_t(x)$, L_t^x are continuous in t and adapted to $\{\mathcal{F}_t\}_{t\in[0,1]}$,
(iii) L_t is non-decreasing with $L_0 = 0$ and

$$\int_0^t \chi_{\{X_s=0\}} dL_s^x = L_t^x, \tag{5}$$

where $\circ$ denotes Stratonovich integral. Eq.(4) is called the reflected SDE, X is called a reflection process on $\Re_+$. If $\sigma = 1$, the process X is the *reflecting Brownian motion.*

See W. Werner; Lecture Notes in Mathematics,**1613**(1995)37-43; Lions P.L. and Sznitman A.S. : Comm.Pure Appl.Math.**XXXVII**, 511-537(1984); Doney R.A. and Zhang T. : Ann.I.H. Poincare-PR, **41**(2005)107-121.

3 Problems

- **The above known results are in the framework of non-anticipating stochastic calculus**
- **What happen if we work on the framework of anticipating stochastic calculus ?**

More precisely, if $X_t = \int_0^t u_s dW(s)$ is the Skorohod integral (i.e., u is a non-adapted process w.r.t. $(\mathcal{F}_t)_{t\geq 0}$), then the problems we concern are the following

(i) How to choose a right way establishing the local time $L(t, x, \omega)$ of X ?

(ii) Does the local time $L(t, x, \omega)$ have some properties we are interested in?

(iii) Does the Eq.(4) has a solution if the stochastic integral in Eq.(4) is Skorohod integral or the initial x is replaced by any nonnegative random variable Z?

Method 1

Imkeller and Nualart (See Ann.Prob. Vol.22, 469-493(1994)), Ustunel (Stochastics 36(65-69)(1991)) first established by using integration by parts the local time $L_1(t, x, \omega)$ of X and the Tanaka formula as follows:

$$(X_t - x)^+ = (-x)^+ + \int_0^t I_{[x,+\infty)}(X_s) u_s dW_s + L_1(t, x, \omega). \qquad (6)$$

Noting that the integrand $I_{[x,+\infty)}(X_s)u_s$ in (6) is not Malliavin differentiable and non-adapted we have no way (such as Meyer's inequalities and Malliavin calculus techniques and etc) doing estimates on moments of the Skorohod integral $\int_0^t I_{[x,+\infty)}(X_s)u_s dW_s$, not to say any property we want! Therefore the formula (6) is just a formula and it does not give us any information about the local time $L_1(t, x, \omega)$.

Method 2: A good way

The way is to make the Skorohod integral $\int_0^t I_{[x,+\infty)}(X_s)u_s dW_s$ in (6) has some like-martingale property to avoid the unpleasant fact: the integrand $I_{[x,+\infty)}(X_s)u_s$ in (6) is not Malliavin differentiable. Then we can make use of Ito and Malliavin Calculus techniques. The important factor of the way is the following Ocone-Clark formula (See Proposition A.1, PTRF 78, 535-581(1988)):

$$F = \mathbf{E}(F|\mathcal{F}_{[s,t]^c}) + \int_s^t \mathbf{E}(\mathcal{D}_\alpha F|\mathcal{F}_{[\alpha,t]^c})dW(\alpha), \quad \text{for } F \in \mathbb{D}^{1,2}. \tag{7}$$

4 Representations for Indefinite Skorohod Integrals

Representations Theorem

Let $X_t = \int_0^t u_s dW(s)$ be the Skorohod integral process, and u belong to the Sobolev space $\mathbb{L}^{k,p} = \mathbb{L}^p([0,1], \mathbb{D}^{k,p})$ for $k \geq 3$, $p > 2$. Then there exists a unique process $v \in \mathbb{L}^{k-2,p}$ such that $X_t = \int_0^t \mathbf{E}[v_s | \mathcal{F}_{[s,t]^c}] dW_s$ for every $t \in [0,1]$. Moreover, $v_\cdot = u_\cdot + \int_0^\cdot D_\cdot u_s dW_s$.

Ito-Skorohod Integral processes

For every $\lambda \leq t$ and $f \in L^2([0,1] \times \Omega)$ we define Y_t^λ by $Y_t^\lambda = \int_0^\lambda \mathbf{E}[f_s | \mathcal{F}_{[s,t]^c}] dW_s$. Then for any fixed $t \in [0,1]$, the process $(Y_t^\lambda)_{\lambda \in [0,t]}$ is an $\mathcal{F}_{(\lambda,t]^c}$- martingale and we have $\lim_{\lambda \uparrow t} Y_t^\lambda = Y_t$ almost surely and in L^2, where $Y_t = \int_0^t \mathbf{E}[f_s | \mathcal{F}_{[s,t]^c}] dW_s$ is called Ito-Skorohod Integral processes if $f \in \mathbb{L}^{k,p} (k \geq 1, p \geq 2)$.

See Tudor et all, Bernoulli 10(2004)313-325, Martingale structure of Skorohod Integral processes, in press in Ann.Prob.(2006).

5 Main Results of Anticipating Local Times

Existence of Anticipating Local Times $L^X(t,x,\omega)$ for the Skorohod integral process

Let $X_t = \int_0^t u_s dW(s)$ be the Skorohod integral process, and u belong to the Sobolev space $\mathbb{L}^{k,p} = \mathbb{L}^p([0,1],\mathbb{D}^{k,p})$ for $k \geq 3$, $p > 2$. Then by the Representations Theorem we have the following Tanaka formula and Occupation times formula for the Skorohod integral process X

$$(X_t - x)^+ = (-x)^+ + \int_0^t I_{[x,+\infty)}(X_t^s)\mathbf{E}\big[v_s|\mathcal{F}_{[s,t]^c}\big]dW_s + \frac{1}{2}L^X(t,x,\omega),$$

$$\int_0^t \Phi(X_t^s)\big(\mathbf{E}[v_s|\mathcal{F}_{[s,t]^c}]\big)^2 ds = \int_{-\infty}^{\infty}\Phi(x)L^X(t,x,\omega)dx$$

where $v_s = u_s + \int_0^s D_s u_t dW_t$, $s \in [0,1]$, $X_t^s = \int_0^s \mathbf{E}[v_r|\mathcal{F}_{[r,t]^c}]dW_r$ ($s \leq t$) is the $It\hat{o}$-Skorohod integral of v, and D is the Malliavin derivative, $L^X(t,x,\omega)$ is called the Anticipating Local Times of Indefinite Skorohod Integrals X .

Comparisons for $L^X(t,x,\omega)$, $L_1(t,x,\omega)$ and $L(t,x,\omega)$

The Local times $L^X(t,x,\omega)$ and $L_1(t,x,\omega)$ do not coincide in general,

they coincide only if the integrand u is adapted and are equal to $L(t,x,\omega)$.

But the Anticipating Local time $L^X(t,x,\omega)$ has the properties we expect.

6 Conclusions

Fractional smoothness for $L^X(t,x,\omega)$ w.r.t. spatial variable x

Theorem 1. Assume that the anticipating integrand u satisfies
(C1) $u \in \mathbb{L}^{k,p}$ and for $k \geq 3$ and any $p \geq 2$,

$$\int_0^1 \mathbf{E}|u_s|^p ds + \int_0^1\int_0^1 \mathbf{E}|D_s u_r|^p ds dr + \int_0^1\int_0^1\int_0^1 \mathbf{E}|D_\alpha D_s u_r|^p d\alpha ds < +\infty.$$

(1) If u satisfies the condition (C1), then for every $t > 0$ and $p \geq 1$ the path $x \longrightarrow L^X(t,x,\omega)$ almost surely belongs to the Besov space $\mathcal{B}^{\frac{1}{2}}_{p,\infty}$.
(2) If u satisfies the condition (C1) and the following condition :
(C2) $\mathbf{E}\big[(u_s + \int_0^s D_s u_r dW_r)\big|\mathcal{F}_s\big] \neq 0$ a.s.(s,ω), $ds \times \mathbf{P}$ on $[0,t] \times \Omega$ for every $t > 0$. Then the path $x \longrightarrow L^X(t,x,\omega)$ almost surely does not belong to $\mathcal{B}^{\frac{1}{2},0}_{p,\infty}$ for every $t > 0$ and $p \geq 1$.

Example. Let $u_s = W_t W_s$ for any $0 \leq s \leq t \leq 1$, then $D_s u_r = W_r \cdot I_{[0,t]}(s) + W_t \cdot I_{[0,r]}(s)$. Moreover, $\mathbf{E}\big[2\big(u_s + \int_0^s D_s u_r dW_r\big)|\mathcal{F}_s\big] = 3W_s^2 - s \neq 0$ a.s.(s,ω), $ds \times \mathbf{P}$ on $[0,t] \times \Omega$ for every $t > 0$, i.e., the stochastic process u satisfies the condition (C2) above.

See Zongxia, Liang, Besov regularity for the generalized local time of the indefinite Skorohod integral, **Annales de L'Institut Henri Poincare ,43(2007)77-86** .

Fractional smoothness for $L^X(t,x,\omega)$ w.r.t. path ω

Theorem 2. Assume that the anticipating integrand u satisfies $u \in \mathbb{L}^{k,p}$ for $k \geq 3$, $p > 2$ and

$$|||u||| \equiv \int_0^1 \mathbf{E}|u_s|^p ds + \int_0^1 \int_0^1 \mathbf{E}|D_s u_r|^p dr ds + \int_0^1 \int_0^1 \int_0^1 \mathbf{E}|D_\alpha D_s u_r|^p dr ds d\alpha$$

$$+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathbf{E}|D_\beta D_\alpha D_s u_r|^p dr ds d\alpha d\beta < +\infty.$$

Then for every $(t,x) \in [0,1] \times \Re$ the Anticipating Local time $\omega \mapsto L^X(t,x,\omega)$ belongs to the fractional Sobolev spaces $\mathbb{D}^{\alpha,p}$ for $\alpha < \frac{1}{2}$ and $p > 2$. Moreover, the result is optimal if X is Brownian motion or fractional Brownian motion with hurst parameter less than $\frac{1}{2}$.

See Zongxia, Liang, Fractional smoothness for the generalized local time of the indefinite Skorohod integral, **JFA, 239(2006)247-267.**

Variational properties for $L^X(t,x,\omega)$ w.r.t. spatial variable x

Theorem 3. Assume that the anticipating integrand u satisfies $u \in \mathbb{L}^{k,p}$ for all $p \geq 1$ and $k \geq 0$. Then we have

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} (L^X(t, x_{i+1}^n, \omega) - L^X(t, x_i^n, \omega))^2 = 4\int_a^b L^X(t,x,\omega)dx$$

holds (α, p)-q.s. for every $0 < \alpha < \frac{1}{6}$, $p > 1$, where $\triangle_n = (x_i^n, x_{i+1}^n)$ be a sequence of subdivisions of $[a,b]$ with $x_i^n = i(b-a)/2^n + a$, $i = 0, 1, \cdots 2^n$.

See Guilan, Cao, Kai, He, Zongxia, Liang, Quasi sure analysis of local times of anticipating smooth semimartingales, **Bull.Sci.Math., Vol.131. No.6,** or See doi:10.1016/j.busci.2006.03.012.

Anticipating reflected stochastic differential equations

Theorem 4. Assume that the function σ, its derivatives σ' and σ'' are Lipschitz continuous, Z is any nonnegative random variable with $\mathbf{P}\big(Z \in [0, +\infty)\big) = 1$. Then there is a pair $(X_t, L_t, t \in [0,1])$ to solve the following anticipating reflected SDE,

$$X_t(Z) = Z + \int_0^t \sigma(X_s(Z)) \circ dB_s + L_t^Z, \ \forall\, t \in [0,1], \tag{8}$$

and satisfy
(i)$X_0(Z) = Z$, $X_t(Z) \geq 0$ for $t \in [0,1]$,
(ii) $X_t(Z)$, L_t^Z are continuous,
(iii) L_t^Z is non-decreasing with $L_0^Z = 0$ and

$$\int_0^t \chi_{\{X_s(Z)=0\}} dL_s^Z = L_t^Z. \tag{9}$$

See Zongxia, Liang, Tusheng, Zhang: Anticipating reflected stochastic differential equations, Preprint (2006) or See arXiv:math/0612294v1

Theorem 5. Assume that $\mathcal{O}$ is a smooth bounded open set in $\Re^d$ and there exists a function $\phi \in \mathcal{C}_b^2(\Re^d)$ such that

$$\exists \alpha > 0,\ \forall x \in \partial\mathcal{O},\ \forall \zeta \in \mathbf{n}(x),\ (\nabla\phi(x), \zeta) \leq -\alpha C_0. \tag{10}$$

Then for any random variable Z with $\mathbf{P}\{Z \in \bar{\mathcal{O}}\} = 1$ there exists a pair $(X_t(Z), L_t^Z, t \in [0,1])$ solving the following stochastic differential equation on domain $\mathcal{O}$ with reflecting boundary conditions:

$$X_t(Z) = Z + \int_0^t \sigma(X_s(Z)) \circ dB_s - L_t^Z \tag{11}$$

with $X_t(Z) \in \bar{\mathcal{O}}$, and satisfying
(1) the function $s \mapsto L_s^Z$ with values in $\Re^d$ has bounded variation on $[0,1]$ and $L_0^Z = 0$.
(2)

$$|L^Z|_t = \int_0^t I_{(X_s(Z) \in \partial\mathcal{O})} d|L^Z|_s, \tag{12}$$

$$L_t^Z = \int_0^t \xi(X_s(Z)) d|L^Z|_s, \tag{13}$$

where $\xi(X_s(Z)) \in \mathbf{n}(X_s(Z))$, if the function σ satisfies that σ and $\nabla\sigma$ are bounded, and the following

$$\begin{aligned} &\|\sigma(x)-\sigma(y)\|+\|(\nabla\sigma\cdot\sigma)(x)-(\nabla\sigma\cdot\sigma)(y)\| \\ &\|(\nabla\sigma\cdot\nabla\sigma\cdot\sigma)(x)-(\nabla\sigma\cdot\nabla\sigma\cdot\sigma)(y)\| \\ &+\|(\sigma^T\cdot\nabla^2\sigma\cdot\sigma)(x)-(\sigma^T\cdot\nabla^2\sigma\cdot\sigma)(y)\|\le k|x-y| \end{aligned} \tag{14}$$

for some constant $k>0$, where C_0 is given by (10), σ^T denotes transpose of σ, $\nabla\sigma$ and $\nabla^2\sigma$ denote σ's derivatives of first and second order with respect to spatial variable x, respectively.

See Zongxia, Liang, Anticipating Multidimensional SDEs with reflections Preprint (2007) or See arXiv:math/0704.271v1

Key points to prove Theorems 4 and 5: Substitution formulas

$$\int_0^t \sigma(X_s(Z)) \circ dB_s = \int_0^t \sigma(X_s(x)) \circ dB_s \Big|_{x=Z}, \tag{15}$$

$$\int_0^t l(X_s(Z)) dL_s^Z = \int_0^t l(X_s(x)) dL_s^x \Big|_{x=Z} \tag{16}$$

where

$$\int_0^t \sigma(X_s(Z)) \circ dB_s$$
$$:= \lim_{\|\pi\| \to 0} \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left(\int_{t_k}^{t_{k+1}} \sigma(X_s(Z)) ds \right) (B_{t_{k+1}} - B_{t_k}).$$

Main idea

Proving that for any $p > 1$ and $R > 0$ the following holds

$$\lim_{\|\pi\|\to 0} \mathbf{E}\{ \sup_{0\le x\le R} |S_\pi(t,x) - I(t,x)|^p \} = 0,$$

where

$$S_\pi(t,x) := \sum_{k=0}^{n-1} \frac{1}{t_{k+1}-t_k} \left(\int_{t_k}^{t_{k+1}} \sigma(X_s(x))ds \right) (B_{t_{k+1}} - B_{t_k}).$$

1. Liang and Zhang : Moments estimates for one-point and two-point motions.

2. Arnold, L. and Imkeller, P. : Lemma of Garsia, Rodemich and Rumsey on stochastic field. See **SPA 62**, 19-54(1996).

3. Yor, M.: Approximation of Stochastic integral. See **LNM** 561,(1977).

4. Zongxia, Liang : Spatial asymptotic behavior of homeomorphic global flows for non-Lipschitz SDEs. In press in **Bull.Sci.math.**(2007), See doi:10.1016/j.bulsci.2006.12.001.

Substitution formula(Method 1)

D.Nualart and P.Malliavin

1. Sobolev Embedding Theorem

$$\sup_{x\in[0,R]} |f_n(x)|^p \le C(p,R)\int_{[0,R]} \left[|f_n(x)|^p + |\frac{\partial f_n(x)}{\partial x}|^p\right] dx$$

2. $\sup_{x\in[0,R]} \mathbf{E}\{|f_n(x)|^p\}$, $\sup_{x\in[0,R]} \mathbf{E}\{|\frac{\partial f_n(x)}{\partial x}|^p\}$.

3.H_1-H_6 in Theorem 3.2.6(See D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, 1995).

Substitution formula(Method 2)

Z.Liang and T. Zhang

1.

$$\sup_{x\in[0,R]}\{|f_n(x)|\} \le \sup_{x\in[0,R]}\{|f_n(x)-f_n(x_0)|\}+|f_n(x_0)|$$
$$\le \sup_{x,y\in[0,R]}\{|f_n(x)-f_n(y)|\}+|f_n(x_0)|.$$

2. Lemma of Garsia, Rodemich and Rumsey (Kolmogorov Lemma)

$$\mathbf{E}\{\sup_{x,y\in[0,R]}|f_n(x)-f_n(y)|^p\}$$
$$\le C(p,R)\int\int_{[0,R]^2}\frac{\mathbf{E}\{|f_n(x)-f_n(y)|^p\}}{d(x,y)^p}m(dx)m(dy).$$

3. $\sup_{n\ge 1}\mathbf{E}\{|f_n(x)-f_n(y)|^p\}\le C(p,R)d(x,y)^p$, $\limsup_{n\to\infty}\sup_{x\in[0,R]}\mathbf{E}\{|f_n(x)|^p\}=0.$

Large Deviations Principles for the Solution $\{X_t(Z), t \in [0,1]\}$

Theorem 6. Assume the same conditions as in Theorem 6. Then for any random variable Z^ε with $\mathbf{P}\{Z^\varepsilon \in \bar{\mathcal{O}}\} = 1$ and the family $\{Z^\varepsilon, \varepsilon > 0\}$ satisfies for $x_0 \in \bar{\mathcal{O}}$ and any $\delta > 0$

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\{|Z^\varepsilon - x_0| > \delta\} = -\infty,$$

the processes $\{X_t^\varepsilon(Z^\varepsilon) : \varepsilon > 0\}$ satisfy the large deviation principle on E with good rate function I^{x_0}. In other words, for any open set G and any closed set F of E, we have

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\{X_\cdot^\varepsilon(Z^\varepsilon) \in G\} \geq -\inf_{f \in G}\{I^{x_0}(f)\},$$
$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}\{X_\cdot^\varepsilon(Z^\varepsilon) \in F\} \leq -\inf_{f \in F}\{I^{x_0}(f)\},$$

where $I^x(f)$ is defined by

$$I^x(f) = \limsup_{y\to x}\{I_2^y(f)\},$$
$$I_2^x(f) = \inf\{I_1(\psi) : f = z^\psi(x)\ \},$$

$$I_1(g) = \begin{cases} \frac{1}{2}\int_0^1 |g(s)|^2 ds, & g \in L^2([0,1];\Re^d), \\ +\infty, & \text{otherwise} \end{cases}$$

for $x \in \bar{\mathcal{O}}$ and $f \in E = C([0,1];\bar{O})$. $z^\psi(x)$ satisfies the following

$$\begin{cases} z_t^\psi(x) = x + \int_0^t b(z_s^\psi(x))ds + \int_0^t \sigma(z_s^\psi(x))\psi(s)ds - k_t^\psi(x), \\ k_t^\psi(x) = \int_0^t \xi(z_s^\psi(x))d|k^\psi(x)|_s, \\ |k^\psi(x)|_t = \int_0^t I_{\{s: z_s^\psi(x)\in\partial\mathcal{O}\}} d|k^\psi(x)|_s \end{cases}$$

where $\psi \in L^2([0,1];\Re^d)$.

See Zongxia, Liang : Large deviations for multidimensional SDEs with reflection. arXiv: math/0705.0405v1.

7 Open Problems

Problem 1

Let B_t^H be a fractional Brownian motion with hurst parameter $H \in (0,1)$, $L_t^{B^H}$ its anticipating local time. $S_t^H = \sup_{s\in[0,t]}\{B_t^H\}$.

Does the stochastic processes $(S_t^H - B_t^H, S_t^H)$ and $(|B_t^H|, L_t^H)$ have the same law?

Remark: $H = \frac{1}{2}$, it just is the Lévy's theorem.

See Theorem 2.3, Chapter VI in D. Revuz, M. Yor ; Continuous Martinglaes and Brownian motion. Springer-Verlag.

Problem 2

Does exist a pair $(X_t(Z), L_t^Z, t \in [0,1])$ to solve the following SDE,

$$\begin{aligned} X_t(Z) &= Z + \int_0^t \sigma(X_s(Z)) \circ dB_s + \alpha \inf_{s\le t}\{X_s(Z)\} \\ &\quad + \beta \sup_{s\le t}\{X_s(Z)\} + L_t^Z, \ \forall\, t \in [0,1], \end{aligned}$$

and satisfy
(i)$X_0(Z) = Z$, $X_t(Z) \geq 0$ for $t \in [0,1]$,
(ii) $X_t(Z)$, L_t^Z are continuous,
(iii) L_t^Z is non-decreasing with $L_0^Z = 0$ and

$$\int_0^t \chi_{\{X_s(Z)=0\}} dL_s^Z = L_t^Z$$

where $|\alpha| < 1$,$|\beta| < 1$?

See Perman, Werner, PTRF108,357-383(1997);

Chaumont, Doney, PTRF 113, 519-534(1999) for $\sigma = 1$ and $Z = 0$.

8 Basic Tools

- **Malliavin Calculus, Meyer's inequalities,Ocone-Clark formula**
  see D.Nualart, The Malliavin Calculus and Related Topics. Springer-Verlag, 1995.
- **Ito Calculus, B-D-G inequalities, theory about the local time established by M. Barlow and M.Yor**
  see D.Revuz, M.Yor, Continuous Martingales and Brownian Motion, Springer-Verlag, 1980.
- **Ciesielski's characterization of the Besov regularity and Schauder basis**
  see Studia Math.107-204(1993).
- **Watanabe's characterization about Sobolev spaces $\mathbb{D}^{\alpha,p}$ and the interpolation theory on Wiener space**
  see PTRF 95(1993)175-198.
- **Garsia-Rodemich-Rumsey lemma and Kolmogorov Lemma**
  see JFA49,198-229(1982), SPA 62, 19-54(1996).

9 Remarks

$$X_t = \int_0^t \mathbf{E}[v_s|\mathcal{F}_{[s,t]^c}]dW_s.$$

$$v_\cdot = u_\cdot + \int_0^\cdot D_\cdot u_s dW_s.$$

$$L^X(t,x,\omega) = \lim_{\lambda\uparrow t}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^\lambda I_{(x-\varepsilon,x+\varepsilon)}(X_t^s)\big(\mathbf{E}[v_s|\mathcal{F}_{[s,t]^c}]\big)^2 ds.\ a.s.$$

$$X_t^\lambda = \int_0^\lambda \mathbf{E}[v_s|\mathcal{F}_{[s,t]^c}]dW_s,\ \lambda \le t.$$

$$\int_0^t f(s)dB_s := \lim_{\|\pi\|\to 0}\sum_{k=0}^{n-1}\frac{1}{t_{k+1}-t_k}\left(\int_{t_k}^{t_{k+1}}\mathbf{E}\big(f(s)\big|\mathcal{F}_{[t_k,t_{k+1}]^c}\big)ds\right)(B_{t_{k+1}}-B_{t_k}).$$

$$\int_0^t f(s)\circ dB_s := \lim_{\|\pi\|\to 0}\sum_{k=0}^{n-1}\frac{1}{t_{k+1}-t_k}\left(\int_{t_k}^{t_{k+1}}f(s)ds\right)(B_{t_{k+1}}-B_{t_k}).$$

Thank You!