Transport Inequalities For Markov Processes

C. Léonard

A joint work with Arnaud Guillin, Liming Wu and Nian Yao Université Paris 10 & École Polytechnique

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Outline



- Spectral gap in C⁰_{Lip}
- Lyapunov function condition

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Outline



Setting And Definitions

- The Markov Process
- Optimal Transport
- Inequalities
- A Connection With Large deviations
 - Large Deviations Of The Occupation Measure
 - The Basic Result
 - Tensorization

Criteria

- Poincaré and log-Sobolev
- Poincaré implies Pinsker
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A Connection With Large deviations

The Markov Process Optimal Transport Inequalities

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Criteria

The Markov Process Optimal Transport Inequalities

The Markov process

Consider

- State space: X is Polish
- Invariant probability measure: $\mu \in \mathcal{P}_{\mathcal{X}}$
- Markov generator: L
 - \mathcal{L} self-adjoint in $L^2(\mu)$
 - the semigroup $(P_t = e^{t\mathcal{L}})_{t\geq 0}$ is μ -ergodic :

$$P_t f = f, \mu$$
-a.e., $\forall t \Rightarrow f = c, \mu$ -a.e.

• Dirichlet form:

$$\mathcal{E}(g):=\langle -\mathcal{L}g,g
angle_{L^2(\mu)}, \quad g\in\mathbb{D}_2(\mathcal{L})\subset L^2(\mu).$$

 \mathcal{E} is closable in $L^2(\mu)$ and $\mathbb{D}(\mathcal{E}) = \mathbb{D}_2(\sqrt{-\mathcal{L}}) \subset L^2(\mu)$.

• Fisher-Donsker-Varadhan information

$$I(\nu|\mu) = \begin{cases} \mathcal{E}\left(\sqrt{\frac{d\nu}{d\mu}}\right), & \text{if } \sqrt{\frac{d\nu}{d\mu}} \in \mathbb{D}(\mathcal{E}), \\ +\infty, & \text{otherwise.} \end{cases}$$

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The Markov Process Optimal Transport Inequalities

Optimal transport cost

Let $\nu, \mu \in \mathcal{P}_{\mathcal{X}}$ be probability measures on \mathcal{X} .

Definition (Optimal transport cost)

$$\mathcal{T}_{c}(\nu,\mu) = \inf_{P(\nu,\mu)} \iint_{\mathcal{X}^{2}} c(x,y) \pi(dxdy)$$

•
$$P(\nu, \mu) = \{ \pi \in \mathcal{P}_{\chi^2}; \pi_1 = \nu, \pi_2 = \mu \}$$

• c(x, y) is a $[0, \infty)$ -valued lower semicontinuous function

Theorem (Kantorovich duality)

$$\mathcal{T}_{c}(\nu,\mu) = \sup_{(u,v)\in\Phi_{c}} \{\nu(u) - \mu(v)\}$$

• $\Phi_c := \{(u, v) \text{ functions}; u(x) - v(y) \le c(x, y), \forall x, y \in \mathcal{X}\}.$

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The Markov Process Optimal Transport Inequalities

Optimal transport cost

Definition (Wasserstein's metric)

Let *d* be a lower semicontinuous metric on \mathcal{X} , the cost $c(x, y) = d^{p}(x, y)$ with $1 \le p < \infty$, gives

$$W_p(\mu,
u) := \mathcal{T}_{d^p}(\mu,
u)^{1/p}$$

Result: It is a metric on (a subset of) $\mathcal{P}_{\mathcal{X}}$. With c = d we have $\mathcal{T}_c = W_1$.

Theorem (Kantorovich-Rubinstein)

$$W_1(\nu,\mu) = \sup \{\nu(u) - \mu(u); \|u\|_{Lip} \le 1\} := \|\nu - \mu\|_{Lip}^*$$

where
$$||u||_{\text{Lip}} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x,y)}$$

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The Markov Process Optimal Transport Inequalities

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The generalized transport cost

Definition (The generalized transport cost)

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• $\Phi = \Phi_c = \{(u, v) \text{ functions}; u(x) - v(y) \le c(x, y), \forall x, y \in \mathcal{X}\}, \text{ gives } \mathcal{T}_{\Phi} = \mathcal{T}_c.$

• $\Phi = \Phi_{\|\cdot\|} := \{(u, u); u : \mathcal{X} \to \mathbb{R}, \|u\| \le 1\}, \text{ gives } \mathcal{I}_{\Phi}(\nu, \mu) = \|\nu - \mu\|^*.$

Example

With $\Phi = \Phi_d$ or $\Phi = \Phi_{\|\cdot\|_{\text{Lip}}}$

 $\mathcal{T}_{\Phi}(\nu,\mu) = W_1(\nu,\mu) = \|\nu-\mu\|_{\mathrm{Lip}}^*.$

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• $\Phi = \Phi_{\|\cdot\|} := \{(u, u); u : \mathcal{X} \to \mathbb{R}, \|u\| \le 1\}, \text{ gives } \mathcal{I}_{\Phi}(\nu, \mu) = \|\nu - \mu\|^*.$

Example

With $\Phi = \Phi_d$ or $\Phi = \Phi_{\|\cdot\|_{\text{Lie}}}$

 $\mathcal{T}_{\Phi}(\nu,\mu) = W_1(\nu,\mu) = \|\nu-\mu\|_{\mathrm{Lip}}^*.$

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The Markov Process Optimal Transport Inequalities

The generalized transport cost

Definition (The generalized transport cost)

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Image: A matrix

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The Markov Process Optimal Transport Inequalities

Transport-Information Inequalities

Definition (The class C)

The function $\alpha : [0, \infty] \mapsto [0, \infty]$ is in the class C if it is convex increasing left continuous and $\alpha(0) = 0$.

Definition (Transport-Information Inequality)

The probability measure μ satisfies $T_{\Phi}I(\alpha)$ with $\alpha \in C$ if

 $\alpha(\mathcal{T}_{\Phi}(\nu,\mu)) \leq l(\nu|\mu), \text{ for all } \nu \in \mathcal{P}_{\mathcal{X}}$

$T_{\Phi}l(\alpha)$

Example $(W_1 I, W_2 I)$

The probability measure μ satisfies $W_1 I(c)$ or $W_2 I(c)$ if

	$W_1^2(u,\mu)$	$4c^2 l(u \mu), \ \forall u$	$W_1 I(c)$
	$W^2_2(u,\mu)$	$4c^2 I(u \mu), \ \forall u$	$W_2 I(c)$

Corresponds to $\alpha_1(r) = r^2/(4c^2)$ and $\alpha_2(r) = r/(4c^2)$

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The Markov Process Optimal Transport Inequalities

Transport-Entropy Inequalities

• Relative entropy:
$$H(\nu|\mu) = \int_{\mathcal{X}} \log\left(\frac{d\nu}{d\mu}\right) d\nu.$$

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Example ([
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• W_2H is Talagrand's T_2 inequality.

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Outline



- Spectral gap in C⁰_{Lip}
- Lyapunov function condition

Large Deviations Of The Occupation Measure The Basic Result Tensorization

The occupation measure

The main actor is the random probability measure

Definition (Occupation measure of the process X)

$$L_t := \frac{1}{t} \int_0^t \delta_{X_s} \, ds, \quad t \ge 0$$

 $L_t(A)$ = ratio of time spent by X in A during [0, t], $A \subset \mathcal{X}$.

Let

- $\beta = \text{Law}(X_0)$ is the initial law and
- \mathbb{P}_{β} is the corresponding law of $(X_t)_{t\geq 0}$.

Theorem (Ergodic theorem)

 $\lim_{t \to \infty} L_t = \mu, \ \mathbb{P}_{\beta}\text{-a.s. with respect to } \sigma(\mathcal{P}_{\mathcal{X}}, \mathcal{B}_{\mathcal{X}}).$

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Under mixing assumptions,

 $\{L_t\}$ obeys the Large Deviation Principle as $t \to \infty$:

$$\mathbb{P}_{\beta}(L_t \in A) \underset{t \to \infty}{\asymp} \exp\Big(-t \inf_{\nu \in A} I(\nu | \mu)\Big), \quad A \subset \mathcal{P}_{\mathcal{X}}$$

"uniformly" in the initial law β .

Remark: $I(\nu|\mu) = 0$ iff $\nu = \mu$.

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Large deviations of $\{L_t\}_{t\geq 0}$

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Large deviations of $\{L_t\}_{t\geq 0}$

Theorem (Wu Liming)

Assume that $\beta \prec \mu$ with $d\beta/d\mu \in L^2(\mu)$. $\{L_t\}_{t\geq 0}$ obeys the LDP in $\mathcal{P}_{\mathcal{X}}$ with rate function $\nu \mapsto I(\nu|\mu)$: For all measurable subset A of $\mathcal{P}_{\mathcal{X}}$,

$$\begin{split} -\inf_{\nu\in \text{inf }A}I(\nu|\mu) &\leq \qquad \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}_{\beta}(L_t\in\mathcal{A})\\ &\leq \qquad \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_{\beta}(L_t\in\mathcal{A}) \leq \qquad -\inf_{\nu\in \text{cl }A}I(\nu|\mu) \end{split}$$

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- Remark: $\mathbb{P}_{\beta} = \frac{d\beta}{d\mu}(X_0)\mathbb{P}_{\mu}$.
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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Large deviations of $\{L_t\}_{t\geq 0}$

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Outline



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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Statement of the basic result

For $\alpha \in C$, define: $\alpha^{\circledast}(\lambda) := \sup_{r \ge 0} \{r\lambda - \alpha(r)\}, \quad \lambda \ge 0.$

Theorem

The following statements are equivalent.

(a)
$$\alpha(\mathcal{T}_{\Phi}(\nu,\mu)) \leq l(\nu|\mu), \quad \forall \nu \in \mathcal{P}_{\mathcal{X}}.$$

(b) $\forall \lambda \geq 0, \forall (u, v) \in \Phi$,

$$\limsup_{t \to \infty} rac{1}{t} \log \mathbb{E}_{\mu} \exp \left(\lambda \int_0^t u(X_s) \, ds
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(c) $\forall r, t > 0, \forall (u, v) \in \Phi, \forall \beta \in \mathcal{P}_{\mathcal{X}} \text{ such that } d\beta/d\mu \in L^{2}(\mu),$

$$\mathbb{P}_{\beta}\left(\frac{1}{t}\int_{0}^{t}u(X_{s})\,ds\geq\mu(v)+r\right)\leq\left\|\frac{d\beta}{d\mu}\right\|_{L^{2}(\mu)}e^{-t\alpha(r)}$$

• Remark: $\frac{1}{t} \int_0^t u(X_s) ds = \int_X u dL_t = L_t(u)$. • • • • • • • • • • • • • • •

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

An immediate corollary

Specializing with W_1 one obtains the

Corollary

The following statements are equivalent.

- (a) $W_1^2(\nu,\mu) \le 4c^2 l(\nu|\mu), \quad \forall \nu;$
- (b) $\forall \lambda \geq 0, \forall u \in C_{\text{Lip}} : \|u\|_{\text{Lip}} \leq 1, \mu(u) = 0,$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\mu}\exp\left(\lambda\int_{0}^{t}u(X_{s})\,ds\right)\leq c^{2}\lambda^{2}$$

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- Interest: Estimation of $\mu(u)$ by $L_t(u)$.
- Analogous result for W₂I.

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Deviation functions, Transport functions

Definition (Transport function)

A function α in C is a transport function if μ satisfies $T_{\Phi}I(\alpha)$

 $\alpha(\mathcal{T}_{\Phi}(\mu,\nu)) \leq I(\nu|\mu), \text{ for all } \nu$

Denote

$$\mathcal{T}(L_t) = \mathcal{T}_{\Phi}(L_t, \mu)$$

Definition (Deviation function)

A function α in C is a deviation function if for all $r \ge 0$

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}_{\beta}(\mathcal{T}(L_t)\geq r)\leq -\alpha(r)$$

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

A deviation function is a transport function

Borrowed from [Gozlan-Léonard].

Recipe

Any deviation function α is a transport function.

ldea of proof.

Suppose *T* is regular enough for the sets *A_r* = {*v* ∈ *P_X*; *T*(*v*) ≥ *r*} to be good sets. For all *r* ≥ 0,

$$\mathbb{P}_{eta}(\mathcal{T}(L_t) \geq r) = \mathbb{P}_{eta}(L_t \in A_r) symp \exp[-ti(r)]$$

with $i(r) = \inf\{l(\nu|\mu); \nu \in \mathcal{P}_{\mathcal{X}} : \mathcal{T}(\nu) \ge r\}.$

- Let α be a deviation function. Then, $\exp[-ti(r)] \leq \exp[-t\alpha(r)], \forall r \geq 0$ i.e. $\alpha(r) \leq i(r), \forall r \geq 0$.
- Taking $r = \mathcal{T}(\nu)$ leads to: $\alpha(\mathcal{T}(\nu)) \leq i(\mathcal{T}(\nu)) \leq l(\nu|\mu), \forall \nu \in \mathcal{P}_{\mathcal{X}}.$

Mostly based on the LD lower bound

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

Outline

Setting And Definitions
The Markov Process
Optimal Transport
Inequalities
A Connection With Large deviations
Large Deviations Of The Occupation Measure
The Basic Result
Tensorization

Criteria

- Poincaré and log-Sobolev
- Poincaré implies Pinsker
- Spectral gap in C⁰_{Lip}
- Lyapunov function condition

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

$T_c I(\alpha)$ for product measures

The ingredients

- $(\mathcal{X}_1, \mu_1), (\mathcal{X}_2, \mu_2), (\mathcal{X}_1 \times \mathcal{X}_2, \mu_1 \otimes \mu_2)$
- $C_1, C_2, C_1 \oplus C_2((X_1, X_2), (y_1, y_2)) = C_1(X_1, y_1) + C_2(X_2, y_2)$

Theorem (Tensorization)

Suppose that the following T_cIs hold:

 $\begin{aligned} &\alpha_1(\mathcal{T}_{c_1}(\nu_1,\mu_1)) \le l(\nu_1 \mid \mu_1), \; \forall \nu_1 \in \mathcal{P}(\mathcal{X}_1) \\ &\alpha_2(\mathcal{T}_{c_2}(\nu_2,\mu_2)) \le l(\nu_2 \mid \mu_2), \; \forall \nu_2 \in \mathcal{P}(\mathcal{X}_2) \end{aligned}$

Then, on $\mathcal{X}_1 \times \mathcal{X}_2$, we have:

 $\alpha_1 \Box \alpha_2 \big(\mathcal{T}_{c_1 \oplus c_2}(\nu, \mu_1 \otimes \mu_2) \big) \le l(\nu \mid \mu_1 \otimes \mu_2), \quad \forall \nu \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$

• $\alpha_1 \Box \alpha_2(r) = \inf \{ \alpha_1(r_1) + \alpha_2(r_2); r_1, r_2 \ge 0 : r_1 + r_2 = r \}$ is the inf-convolution of α_1 and α_2 .

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

$T_c I(\alpha)$ for product measures

Typical LD result

Let $\{Z_t^1\}_t$ and $\{Z_t^2\}_t$ be two *independent* LD systems on \mathbb{R} with respective rate functions α_1 and α_2 . Then, $\{Z_t := Z_t^1 + Z_t^2\}_t$ obeys the LDP with rate function $\alpha = \alpha_1 \Box \alpha_2$.

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

n-Tensorization

On \mathcal{X}^n define the sum-cost

• $\bigoplus c(x_1,\ldots,x_n;y_1,\ldots,y_n) := \sum_{1 \le i \le n} c(x_i;y_i), \quad x_1,\cdots,y_n \in \mathcal{X}$

and the sum-Dirichlet form

• $\bigoplus \mathcal{E}(g) := \int_{\mathcal{X}^n} \sum_{1 \le i \le n} \mathcal{E}(g_i^{\hat{x}_i}) \prod_{1 \le i \le n} \mu(dx_i)$

where $g_i^{x_i}(x_i) = g(x_1, ..., x_n)$ with $\hat{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

It is the Dirichlet form of $(X_t^1, \dots, X_t^n)_{t \ge 0}$ with X^1, \dots, X^n independent copies of *X*.

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

n-Tensorization

Theorem

If μ satisfies $T_c I$, then $\mu^{\otimes n}$ satisfies

$$n\alpha\left(\frac{T_{\oplus c}(\nu,\mu^{\otimes n})}{n}\right) \leq I_{\oplus \mathcal{E}}(\nu|\mu^{\otimes n}), \quad \forall \nu \in \mathcal{P}(\mathcal{X}^n)$$

In particular, for all $(u, v) \in \Phi_c$, $\beta \in \mathcal{P}(\mathcal{X}^n)$, t, r > 0,

$$\mathbb{P}_{\beta}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{t}\int_{0}^{t}u(X_{s}^{i})\,ds\geq\mu(v)+r\right)\leq\left\|\frac{d\beta}{d\mu^{\otimes n}}\right\|_{L^{2}(\mu^{\otimes n})}e^{-nt\alpha(r)}.$$

Proof.

$$\alpha_n(r) = \Box^n \alpha(r) = n \alpha(r/n), r \ge 0.$$

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Large Deviations Of The Occupation Measure The Basic Result Tensorization

n-Tensorization

Example $(W_1 I)$

As $W_1 I(c) : W_1^2(\nu, \mu) \le 4c^2 I(\nu|\mu)$ corresponds to $\alpha_1(r) = r^2/(4c^2)$ we have $W_1^2(\nu, \mu^{\otimes n}) \le 4c^2 n I(\nu|\mu^{\otimes n})$

Example $(W_2 I)$

As $W_2 I(c) : W_2(\nu, \mu) \le 4c^2 I(\nu|\mu)$ corresponds to $\alpha_2(r) = r/(4c^2)$ we have $W_2(\nu, \mu^{\otimes n}) \le 4c^2 I(\nu|\mu^{\otimes n})$

- The constant $4c^2n$ in W_1I explodes as n.
- The constant is *dimension-free* in W_2I .
- Beautiful applications of W₂I for Gibbs measures in situations where log-Sobolev is unknown (Gao and Wu, 2007).

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C^0_{Lip} Lyapunov function condition

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C^0_{Lip} Lyapunov function condition

Poincaré and log-Sobolev

Definition (Poincaré inequality) $\operatorname{Var}_{\mu}\left(\sqrt{\frac{d\nu}{d\mu}}\right) \leq cl(\nu|\mu), \quad \forall \nu$ P(c)

Definition (log-Sobolev inequality)

 $H(\nu|\mu) \leq 2cl(\nu|\mu), \quad \forall \nu$

LS(c)

Theorem

$$C_P(\mu) \leq C_{W_2I}(\mu) \leq C_{LS}(\mu).$$

Proof.

Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C^0_{Lip} Lyapunov function condition

Poincaré and log-Sobolev

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 - Poincaré implies Pinsker
 - Spectral gap in C⁰_{Lip}
 - Lyapunov function condition

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C^0_{Lip} Lyapunov function condition

Poincaré implies a Pinsker type inequality

Theorem

If μ satisfies P(c), then

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In particular, $\forall t, r, \beta, u$,

$$\mathbb{P}_{\beta}\left(\frac{1}{t}\int_{0}^{t}u(X_{s})\,ds\geq\mu(u)+r\right)\leq\left\|\frac{d\beta}{d\mu}\right\|_{L^{2}(\mu)}\exp\left(-\frac{tr^{2}}{c\delta(u)^{2}}\right)$$

where $\delta(u) := \sup(u) - \inf(u)$.

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- $d(x,y) = 2_{x \neq y}$ so that: $W_1(\nu,\mu) = \|\nu \mu\|_{TV}$;
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 - (M)_{P20} a Poisson(1) process;
 - (Ya) and Windependent:
 - $X_{t} := Y_{N_{t}} t \ge 0$
 - so that: $I(\nu|\mu) = \operatorname{Var}_{\mu}\left(\sqrt{\frac{d\nu}{d\mu}}\right), \forall i$

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C⁰_{Lip} Lyapunov function condition

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C⁰_{Lip} Lyapunov function condition

Spectral gap in C_{Lip}^0

Definition (Spectral gap in C_{Lip}^{0})

There exists $c < \infty$ such that for all $g \in C_{Lip} \cap L^2_0(\mu)$, there exists $f \in L^2_0(\mu)$ such that

•
$$-\mathcal{L}f = g$$

•
$$\|\tilde{f}\|_{\text{Lip}} \leq c \|g\|_{\text{Lip}}, \quad \tilde{f} : \text{version of } f.$$

•
$$gap(-\mathcal{L}) = 1/c$$
.

Theorem

With $\mathcal{L} = \Delta - \nabla V \cdot \nabla$, the spectral gap in C_{Lip}^0 implies $W_1 I(c)$.

• Remark:
$$\mu = e^{-V} dx$$
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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C⁰_{Lip} Lyapunov function condition

Spectral gap in C_{Lip}^0

Application

With $\mathcal{L} = \Delta - \nabla V \cdot \nabla$, we have: Ric $+ \nabla^2 V \ge K > 0 \Rightarrow W_1 I(K^{-1})$.

Remark: By Bakry-Emery criterion log-Sob holds. Proof without log-Sob.

Application

With $dX_t = \sqrt{2}\sigma(X_t) dB_t + b(X_t) dt$ and

 $\begin{cases} \text{trace}[(\sigma(y) - \sigma(x))(\sigma(y) - \sigma(x))^T] + \langle y - x, b(y) - b(x) \rangle \leq -\delta |y - x|^2, \forall x, y \\ (P_t) \text{ is symmetric in } L^2(\mu) \end{cases}$

then $W_1 I(||\sigma||_{\infty}/\delta)$ holds true.

- unique invariant measure μ, unknown to be estimated;
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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C⁰_{Lip} Lyapunov function condition

Lyapunov function condition

Definition (Lyapunov function condition)

With $U : \mathcal{X} \rightarrow [1, \infty)$ continuous; $\phi : \mathcal{X} \rightarrow [0, \infty)$; b > 0,

$$-\frac{\mathcal{L}U}{U} \ge \phi - b, \quad \mu ext{-a.e.}$$

Theorem

Under this Lyapunov function condition, if μ satisfies Poincaré inequality, then

$$\|\phi.(\nu-\mu)\|_{\mathrm{TV}} \le C\left(\sqrt{I(\nu|\mu)} + I(\nu|\mu)\right), \quad \forall \nu.$$

•
$$d(x, y) = [\phi(x) + \phi(y)] \mathbf{1}_{x \neq y};$$

• $\alpha(r) = O_{r \simeq 0}(r^2) \text{ and } \alpha(r) = O_{r \simeq +\infty}(r)$

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u|\mu)\right), \quad \forall
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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C⁰_{Lip} Lyapunov function condition

Lyapunov function condition

Definition (Lyapunov function condition)

With $U : \mathcal{X} \to [1, \infty)$ continuous; $\phi : \mathcal{X} \to [0, \infty)$; b > 0,

$$-\frac{\mathcal{L}U}{U} \ge \phi - b, \quad \mu ext{-a.e.}$$

Theorem

Under this Lyapunov function condition, if μ satisfies Poincaré inequality, then

$$\|\phi.(\nu-\mu)\|_{\mathrm{TV}} \leq \mathcal{C}\left(\sqrt{I(\nu|\mu)} + I(\nu|\mu)\right), \quad \forall \nu.$$

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Poincaré and log-Sobolev Poincaré implies Pinsker Spectral gap in C^0_{Lip} Lyapunov function condition

Lyapunov function condition

Corollary

With $\mathcal{L} = \Delta - \nabla V \cdot \nabla : \mu = e^{-V} dx$. If

•
$$|x - x_o|^2 \le c(1 + |\nabla V|^2(x)), \forall x;$$

•
$$\limsup_{|x|\to\infty} \Delta V(x)/|\nabla V|^2(x) < 1$$
,

then $W_1I(1/(4c))$ holds and for all t, r > 0

$$\mathbb{P}_{\beta}\left(\frac{1}{t}\int_{0}^{t}u(X_{s})\,ds\geq\mu(u)+r\right)\leq\left\|\frac{d\beta}{d\mu}\right\|_{L^{2}(\mu)}e^{-tr^{2}/c}.$$

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