Intrinsic Ultracontractivity (IU) for Non-symmetric Lévy Processes

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References

This talk is based on a joint work with Renming Song (University of Illinois at Urbana-Champaign).

KS1 P. Kim and R. Song, Intrinsic Ultracontractivity of Non-symmetric Diffusion Semigroups in Bounded Domains, In preprint.

KS2 P. Kim and R. Song, Intrinsic Ultracontractivity for Non-symmetric Levy Processes, to appear in Math. Forum

http://www.math.snu.ac.kr/∼pkim/preprint.html

Outline

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Setup : IU for Brownian motion

W: Brownian motion
D: a bounded domain in $\mathbb{R}^d$.

$$\tau_D = \inf\{t > 0 : W_t \notin D\}.$$

Let

$$W_t^D(\omega) = \begin{cases} W_t(\omega) & \text{if } t < \tau_D(\omega) \\ \partial & \text{if } t \geq \tau_D(\omega), \end{cases}$$

where ∂ is a coffin state added to $\mathbb{R}^d$.The process W^D, i.e., the process W killed upon leaving D, usually called the killed process in D.

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$\{P_t^D\}$: the semigroup of W^D. i.e.,

$$P_t^D f(x) := \mathbb{E}_x[f(W_t^D)]$$

$p^D(t,x,y)$: Transition density function of W^D. i.e.,

$$P_t^D f(x) := \mathbb{E}_x[f(W_t^D)] = \int_D p^D(t,x,y) f(y) dy.$$

$\phi_0(x)$: the eigenfunction for $\Delta|_D$ corresponding to the largest eigenvalue with $\|\phi_0\|_2 = 1$.

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Definition of IU for Brownian motion

Definition. (Davies-Simon (84)) $\{P_t^D\}$ *is said to be intrinsic ultracontractive if for any $t > 0$, $\exists c_t > 0$ such that*

$$p^D(t, x, y) \le c_t \phi_0(x) \phi_0(y).$$

$\{P_t^D\}$ is IU if D is a bounded Lipschitz domain.

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Known Results on IU for Brownian motion

More generally, $\{P_t^D\}$ is IU if D is one of the following types of bounded domains:

- uniform Hölder domain of order $\alpha \in (0, 2)$. (Banuelos, 91).
- Hölder domain of order 0. (Banuelos, 91).
- twisted Hölder domain of order $\alpha \in (1/3, 1]$, (Bass-Burdzy, 92).
- domains which can be locally represented as the region above the graph of a function. (Bass-Burdzy, 92)

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- Is there a natural extension of the definition of IU to general (non-symmetric) strongly continuous contraction semigroups?
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IU in Non-symmetric case: Setup

- E is locally compact separable metric space. m is positive finite measure on E such that $\mathrm{Supp}[m] = E$.
- Suppose that $\{P_t\}$ and $\{\hat{P}_t\}$ are are strongly continuous semigroups in $L^2(E, m)$, which are dual each other.

- Assume that there exist continuous, strictly positive and bounded transition density functions $\{p(t, \cdot, \cdot) : t > 0\}$ on $E \times E$.

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IU in Non-symmetric case: Setup

Let L and $\hat{L}$ be the generators of $\{P_t\}$ and $\{\hat{P}_t\}$ on $L^2(E,m)$ resp. It follows from Jentzsch's Theorem that the common value $\lambda_0 := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(\hat{L}))$ is an e-value of multiplicity 1 for both L and $\hat{L}$, and that an e-function ϕ_0 of L associated with λ_0 can be chosen to be strictly positive with $\|\phi_0\|_{L^2(E,m)} = 1$ and an e-function ψ_0 of $\hat{L}$ associated with λ_0 can be chosen to be strictly positive with $\|\psi_0\|_{L^2(E,m)} = 1$. Put

$$q(t,x,y) := \frac{e^{-\lambda_0 t}}{\phi_0(x)} p(t,x,y)\phi_0(y)$$

$$\hat{q}(t,x,y) := \frac{e^{-\lambda_0 t}}{\psi_0(y)} p(t,x,y)\psi_0(x).$$

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The operators $\{Q_t\}$ and $\{\hat{Q}_t\}$ defined by

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form semigroups in $L^2(E,m)$ with $Q_t 1 = \hat{Q}_t 1 = 1$.

Define a function $\mu(x)$ by

$$\mu(x) := \frac{\phi_0(x)\psi_0(x)}{\int_E \phi_0(y)\psi_0(y)m(dy)}.$$

μ is an invariant function of $\{Q_t\}$ and $\{\hat{Q}_t\}$. So $\{Q_t\}$ and $\{\hat{Q}_t\}$ are dual semigroups on $L^2(E,\mu(x)m(dx))$. Moreover, they are two strongly continuous contraction semigroups on $L^2(E,\mu(x)m(dx))$.

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Equivalently, the semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ are said to be intrinsic ultracontractive iff, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p(t, x, y) \le c_t \phi_0(x) \psi_0(y), \quad \forall (x, y) \in E \times E.$$

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Outline

(Discontinuous) Lévy Processes

Let $X = (X_t, \mathbb{P}_x)$ be a Lévy process in $\mathbb{R}^d$ with the generating triplet (A, ν, γ).i.e., for every $z \in \mathbb{R}^d$,

$$\mathbb{E}_0 \left[e^{iz\cdot X_1} \right]$$

$$= \exp\left(-\frac{1}{2} z \cdot Az + i\gamma \cdot z + \int_{\mathbb{R}^d} (e^{iz\cdot x} - 1 - iz \cdot x 1_{\{|x|\le 1\}}(x))\nu(dx) \right)$$

where A is a symmetric nonnegative definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$, and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

γ is called the drift of X and ν is called the Lévy measure of X.

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Dual for Lévy Processes

$\widehat{X} := -X$ is also a Lévy process and it is the dual of X. i.e.

$$\int_{\mathbb{R}^d} P_t f(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \widehat{P}_t g(x) dx$$

where

$$P_t f(x) := \mathbb{E}_x[f(X_t)] \quad \text{and} \quad \widehat{P}_t f(x) := \mathbb{E}_x[f(\widehat{X}_t)].$$

Assumption 1 : Assumptions on Lévy measure

The Lévy measure ν satisfies either (a) or (b) below:

A1(a)

There exists Borel function $L(x) \geq 0$ such that $\forall B$,

$$|B| = \int_B L(x)\nu(dx).$$

Moreover, we assume that $L \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$.

A1(b)

Let $M(x)$ be the Radon-Nikodym derivative of the absolutely continuous part of ν. We assume that there exists $R_0 > 0$ such that

$$\inf_{x \in B(0,R_0)} M(x) > 0.$$

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Killed semigroup

D: Open set

For any $t > 0$, define

$$P_t^D f(x) := \mathbb{E}_x[f(X_t^D)] \quad \text{and} \quad \widehat{P}_t^D f(x) := \mathbb{E}_x[f(\widehat{X}_t^D)].$$

The next equality is known as Hunt's switching identity.

$$\int_D f(x) P_t^D g(x) dx = \int_D g(x) \widehat{P}_t^D f(x) dx.$$

For any open set D with finite Lebesgue measure, $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are both strongly continuous contraction semigroups in $L^2(D, dx)$.

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Assumption 2 & 3 : Assumptions on killed density

The next two assumptions are needed to define intrinsic ultracontractivity for non-symmetric semigroups.

(A2)

The transition density function $p^D(t,x,y)$ for X_t^D exists and it is strictly positive. Moreover each $t > 0$, $p^D(t,\cdot,\cdot)$ is continuous in $D \times D$.

We also assume that $p^D(t,\cdot,\cdot)$ is bounded.

(A3)

$\{P_t^D\}$ is ultracontractive. i.e., for $t > 0$, there exists positive constant c_t such that

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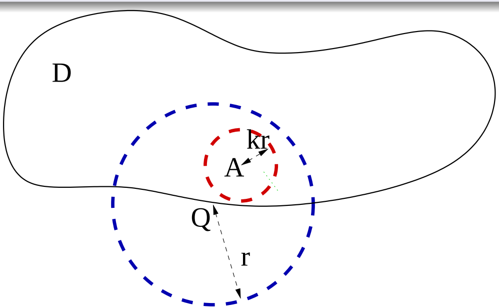
κ-fat open sets

Definition (Song & Wu, 99)

An open set D is κ-fat if there exist $R > 0$ and κ such that for each $Q \in \partial D$ and $r \in (0, R)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (R, κ) is called the characteristics of the κ-fat open set D.



κ-fat open set

Class of smooth domains
$\subset$ Class of Lipschitz domains
$\subset$ Class of non-tangentially accessible domains
$\subset$ Class of uniforms domain
$\subset$ Class of John domains
$\subset$ Class of κ-fat open sets

The boundary of a κ-fat open set can be highly non-rectifiable and, in general, no regularity of its boundary can be inferred. Bounded κ-fat open set can even be locally disconnected.

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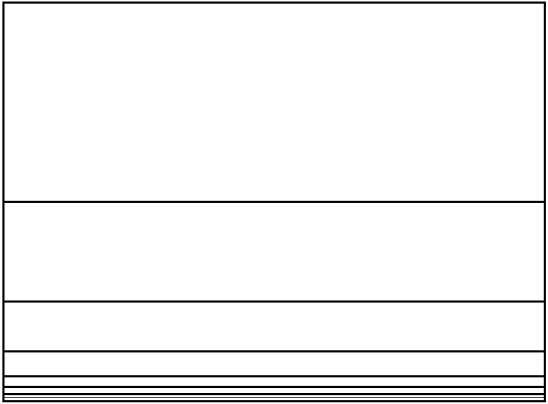
κ-fat open set (an example)



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Depending on whether (A1)(a) or (A1)(b) is valid, our assumptions on the open set D are different.

(A4)(a)

If ν satisfies (A1)(a), we assume that D is an arbitrary bounded open set.

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If ν satisfies (A1)(b), then we assume that D is a bounded κ-fat open set with the characteristics (R, κ).

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Outline

Setup for IU

A_D, $\widehat{A}_D$: L^2 generators of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ respectively.

$\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are compact operators in $L^2(D, dx)$.

Our assumptions, Jentzsch's Theorem (Theorem V.6.6 on page 337 of H. H. Schaefer, Banach lattices and positive operators) and the strong continuity of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$

$\Longrightarrow \lambda_0 := \sup \mathrm{Re}(\sigma(A_D)) = \sup \mathrm{Re}(\sigma(\widehat{A}_D)) < 0$ is an eigenvalue of multiplicity 1 for both A_D and $\widehat{A}_D$, and that an eigenfunction ϕ_0 of A associated with λ_0 can be chosen to be strictly positive a.e. with $\|\phi_0\|_{L^2(D)} = 1$ and an eigenfunction ψ_0 of $\widehat{A}_D$ associated with λ_0 can be chosen to be strictly positive a.e. with $\|\psi_0\|_{L^2(D)} = 1$.

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Setup for IU

By our assumption (continuity and strict positivity of killed density), in fact, $\phi_0(x)$ and $\psi_0(x)$ are strictly positive and continuous in D. Moreover, for every $(x, y) \in D \times D$,

$$
\begin{aligned}
e^{\lambda_0 t}\phi_0(x) &= \int_D p^D(t, x, z)\phi_0(z)dz, \\
-\frac{1}{\lambda_0}\phi_0(x) &= \int_D G_D(x, z)\phi_0(z)dz \\
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Intrinsic ultracontractivity

Definition

The semigroups $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are said to be intrinsic ultracontractive if, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p^D(t, x, y) \le c_t\, \phi_0(x)\, \psi_0(y), \quad \forall (x, y) \in D \times D.$$

Theorem

the semigroup of any killed (non-symmetric) Lévy process X^D satisfying (A1)-(A4) is intrinsic ultracontractive.

Symmetric Lévy process case:
Chen & Song (97, 00), Kulczycki(98), Grzywny(07)

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Observation

By (A3) and the semigroup property, there exists $c_1(t) > 0$ such that

$$\begin{aligned} p^D(t,x,y) &= \int_D p^D(\frac{t}{3},x,z) \int_D p^D(\frac{t}{3},z,w) p^D(\frac{t}{3},w,y) dwdz \\ &\le c_1(t) \int_D p^D(\frac{t}{3},x,z) dz \int_D p^D(\frac{t}{3},w,y) dw \\ &= c_1(t) \mathbb{P}_x(\tau_D > t/3) \mathbb{P}_y(\widehat{\tau}_D > t/3). \end{aligned}$$

By applying Chebyshev's inequality we get

$$p^D(t,x,y) \le \frac{c_1(t)}{9t^2} \mathbb{E}_x[\tau_D] \mathbb{E}_y[\widehat{\tau}_D].$$

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There exist $c_1 > 0$ and $t_0 > 0$ such that for every $t \le t_0$ and $x \in D$

$$\mathbb{P}_x(X_t \in B_2, \tau_D > t) = \int_{B_2} p^D(t,x,z)dz \ge c_1 \int_{D \setminus B_2} G_D(x,y)dy,$$

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The main Lemma does not need the strict positivity of the density of killed processes. In fact, the above lemma can be used to prove the strict positivity of the density of killed processes for some particular (non-symmetric) Lévy processes in disconnected open sets.

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Consequence of Main Lemma

Lemma

There exists a constant $c > 0$ such that

$$\mathbb{E}_x[\tau_D] \le c\,\phi_0(x) \quad \text{and} \quad \mathbb{E}_y[\widehat{\tau}_D] \le c\,\psi_0(y) \qquad \forall (x,y) \in D \times D.$$

Proof. By Main Lemma, there exists $c_1 > 0$ such that

$$\begin{aligned}\mathbb{E}_x[\tau_D] &= \int_{B_2} G_D(x,z)dz + \int_{D\setminus B_2} G_D(x,z)dz \\ &\le \int_{B_2} G_D(x,z)dz + c_1 \int_{B_2} p^D(t_0,x,z)dz.\end{aligned}$$

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Consequence of Main Lemma

Thus we have

$$\int_{B_2} G_D(x,z)dz + c_1 \int_{B_2} p^D(t_0,x,z)dz$$

$$\le c_2 \left(\int_{B_2} G_D(x,z)\phi_0(z)dz + c_1 \int_{B_2} p^D(t_0,x,z)\phi_0(z)dz \right)$$

$$\le c_2 \left(\int_{D} G_D(x,z)\phi_0(z)dz + c_1 \int_{D} p^D(t_0,x,z)\phi_0(z)dz \right)$$

$$= c_2 \left(-\frac{1}{\lambda_0} + c_1 e^{\lambda_0 t_0} \right) \phi_0(x)$$

for some positive constant c_2.

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Proof of Main Lemma (Kulczycki, 98)

- If ν satisfies (A1)(a), then choose a point x_0 in D and $r_0 \in (0, \infty)$ such that $B(x_0, 2r_0) \subset \overline{B(x_0, 2r_0)} \subset D$. We put $B_0 := B(x_0, r_0/2)$, $C_1 := \overline{B(x_0, r_0)}$ and $B_2 := B(x_0, 2r_0)$.
- If ν satisfies (A1)(b) and D is a bounded κ-fat open with the characteristics (R, κ) with $R \le \frac{1}{2} R_0$ where R_0 is the constant in (A1)(b). Let $\rho(x)$ be the distance of a point x to the boundary of D, i.e., $\rho(x) = \text{dist}(x, \partial D)$. Define

$$\begin{aligned} B_0 &:= \{x \in D : \rho(x) > R\kappa/2\}, \\ C_1 &:= \{x \in D : \rho(x) \ge R\kappa/4\}, \\ B_2 &:= \{x \in D : \rho(x) > R\kappa/8\}. \end{aligned}$$

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Proof of Main Lemma

Define

$$\eta_U := \inf\{t \geq 0 : X_t \notin U\}.$$

Let θ be the usual shift operator for Markov processes, and we define stopping times S_n and T_n recursively by

$$S_1 := 0, \quad T_n := S_n + \eta_{D \setminus C_1} \circ \theta_{S_n} \quad \text{and} \quad S_{n+1} := T_n + \eta_{B_2} \circ \theta_{T_n}.$$

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Proof of Main Lemma

There exists a constant $c > 0$ such that for every $x \in D$

$$\mathbb{P}_x(X_{T_n} \in C_1) \geq c\,\mathbb{E}_x[T_n - S_n]. \tag{1}$$

proof of (1) with (A1): $x \in D \setminus C_1$

$$\begin{aligned}
\mathbb{P}_x\left(X_{\eta_{D\setminus C_1}} \in C_1\right) &\geq \mathbb{P}_x\left(X_{\tau_{D\setminus C_1}} \in B_0\right) \\
&= \int_{D\setminus C_1} G_{D\setminus C_1}(x,y) \int_{y-B_0} \nu(dz)dy \\
&\geq \int_{D\setminus C_1} G_{D\setminus C_1}(x,y) \frac{|B_0|^2}{\|1_A L\|^2_{L^2(\nu)}} dy \\
&= \frac{|B_0|^2}{\|1_A L\|^2_{L^2(\nu)}} \mathbb{E}_x\left[\tau_{D\setminus C_1}\right] = \frac{|B_0|^2}{\|1_A L\|^2_{L^2(\nu)}} \mathbb{E}_x\left[\eta_{D\setminus C_1}\right].
\end{aligned}$$

Proof of Main Lemma

Since $T_n = S_n + \eta_{D\setminus C_1} \circ \theta_{S_n}$, by the strong Markov property,

$$\mathbb{P}_x\left(X_{T_n} \in C_1\right) = \mathbb{P}_x\left(X_{S_n+\eta_{D\setminus C_1}\circ\theta_{S_n}} \in C_1\right) = \mathbb{E}_x\left[\mathbb{P}_{X_{S_n}}\left(\eta_{D\setminus C_1} \in C_1\right)\right].$$

So we get

$$\begin{aligned}\mathbb{P}_x\left(X_{T_n} \in C_1\right) &\geq c\,\mathbb{E}_x\left[\mathbb{E}_{X_{S_n}}\left[\eta_{D\setminus C_1}\right]\right] \\ &= c\,\mathbb{E}_x[\eta_{D\setminus C_1} \circ \theta_{S_n}] = c\,\mathbb{E}_x\left[T_n - S_n\right].\end{aligned}$$

□

Proof of Main Lemma

By the separation property for Feller processes, there exists t_0 such that

$$\inf_{y \in C_1} \mathbb{P}_y(\tau_{B_2} > t) \geq \frac{1}{2} \tag{2}$$

for any $t \leq t_0$

Proof of Main Lemma

For $t \le t_0$,

$$\begin{aligned}
\mathbb{P}_x(X_t \in B_2,\ \tau_D > t) &= \mathbb{P}_x(\cup_{n=1}^{\infty}\{X_t \in B_2,\ S_n \le t < S_{n+1}\}) \\
&\ge \mathbb{P}_x(\cup_{n=1}^{\infty}\{X_t \in B_2,\ T_n \le t < S_{n+1}\}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}_x(X_t \in B_2,\ T_n \le t < S_{n+1}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}_x(T_n \le t < S_{n+1}) \\
&= \sum_{n=1}^{\infty} \mathbb{P}_x(T_n \le t < T_n + \eta_{B_2} \circ \theta_{T_n}) \\
&= \sum_{n=1}^{\infty} \mathbb{E}_x\left[\mathbb{P}_{X_{T_n}}(t < \eta_{B_2})\right]
\end{aligned}$$

Proof of Main Lemma

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\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{E}_x \left[\mathbb{P}_{X_{T_n}}(t < \eta_{B_2}) \right] &\geq \sum_{n=1}^{\infty} \mathbb{E}_x \left[\mathbb{P}_{X_{T_n}}(t < \tau_{B_2}) : X_{T_n} \in C_1 \right] \\
&> \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}_x(X_{T_n} \in C_1) \quad (\text{ by (2)}) \\
&\geq c_1 \sum_{n=1}^{\infty} \mathbb{E}_x[T_n - S_n] \quad (\text{ by (1)}) \\
&= c_1 \sum_{n=1}^{\infty} \mathbb{E}_x \left[\int_{S_n}^{T_n} 1_{\mathbb{R}^d}(X_t) dt \right]
\end{aligned}
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\geq \quad & c_1 \mathbb{E}_x \left[\sum_{n=1}^{\infty} \int_{S_n}^{T_n} 1_{D\setminus B_2}(X_t)dt \right] \\
= \quad & \mathbb{E}_x \left[\sum_{n=1}^{\infty} \int_{S_n}^{T_n} 1_{D\setminus B_2}(X_t)dt + \sum_{n=1}^{\infty} \int_{T_n}^{S_{n+1}} 1_{D\setminus B_2}(X_t)dt \right] \\
= \quad & c_1 \mathbb{E}_x \int_0^{\tau_D} 1_{D\setminus B_2}(X_t)dt = c_1 \int_{D\setminus B_2} G_D(x,y)dy.
\end{aligned}
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The upper bound implies the lower bound

$$c_t^{-1}\phi_0(x)\psi_0(y) \le p^D(t,x,y) \le c_t\phi_0(x)\psi_0(y), \quad \forall (x,y) \in D \times D.$$

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Estimates of Green function

There exist constants $c_i > 0, i = 1, 2$ such that

$$c_1\, \mathbb{E}_x[\tau_D]\, \mathbb{E}_y[\widehat{\tau}_D] \le c_2 \phi_0(x)\psi_0(y) \le G_D(x, y), \qquad (x, y) \in D \times D.$$

Moreover, there exists constant $c_3 > 0$ such that

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Convergence to Equilibrium exponentially fast

There exist positive constants c and ν such that for every $(t, x, y) \in (1, \infty) \times D \times D$

$$\left| \left(e^{-\lambda_0 t} \int_D \phi_0(z)\psi_0(z)dz \right) \frac{p^D(t, x, y)}{\phi_0(x)\psi_0(y)} - 1 \right| \leq ce^{-\nu t}.$$

Parabolic boundary Harnack principle

For each positive u there exists $c = c(D, u) > 0$ such that

$$\frac{p^D(t,x,y)}{p^D(t,x,z)} \geq c\,\frac{p^D(s,v,y)}{p^D(s,v,z)}, \quad \frac{p^D(t,y,x)}{p^D(t,z,x)} \geq c\,\frac{p^D(s,y,v)}{p^D(s,z,v)}$$

for every $s, t \geq u$ and $v, x, y, z \in D$.

Expectation of Conditional life time

A Borel function h defined on D is said to be superharmonic with respect to X^D if

$$h(x) \geq \mathbb{E}_x\left[h(X^D_{\tau_B})\right], \qquad x \in B,$$

for every bounded open set B with $\overline{B} \subset D$. We use SH^+ to denote families of nonnegative superharmonic functions of X^D.For any $h \in SH^+$, we use $\mathbb{P}^h_x$ to denote the law of the h-conditioned process X^D and use $\mathbb{E}^h_x$ to denote the expectation with respect to $\mathbb{P}^h_x$.i.e.,

$$\mathbb{E}^h_x\left[g(X^D_t)\right] = \mathbb{E}_x\left[\frac{h(X^D_t)}{h(x)} g(X^D_t)\right].$$

Let ζ^h be the lifetime of the h-conditioned process X^D.

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Expectation of Conditional life time

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$$\sup_{x\in D, h\in SH^+} \mathbb{E}_x^h[\zeta^h] < \infty.$$

(2)

For any $h \in SH^+$, we have

$$\lim_{t\uparrow\infty} e^{-\lambda_0 t}\mathbb{P}_x^h(\zeta^h > t) = \frac{\phi_0(x)}{h(x)} \int_D \psi_0(y)h(y)dy \Big/ \int_D \phi_0(y)\psi_0(y)dy.$$

In particular,

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Outline

Non-symmetric strictly α-stable processes

Let $\alpha \in (0,2)$ and $d \geq 2$. The process X is said to be strictly α-stable if $(X_{at}, \mathbb{P}_0)_{t\geq 0}$ is equal to $(a^{1/\alpha}X_t, \mathbb{P}_0)_{t\geq 0}$ in distribution. We assume that the Lévy measure ν has a density $f(x)$ with respect to the d-dimensional Lebesgue measure, and

$$\kappa\,|x|^{-(d+\alpha)} \leq f(x) \leq \kappa^{-1}\,|x|^{-(d+\alpha)}, \qquad x \in \mathbb{R}^d.$$

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IU for (non-symmetric) strictly α-stable processes

Using our Main Lemma and results in Vondraček (02), Assumptions are true for any bounded open sets (not necessarily connected). So IU is true for any killed non-symmetric strictly α-stable processes if D is bounded.

Proof of strict positivity of killed density

X^D has a continuous transition density $p^D(t,x,y)$. By Theorem 3.2 in Vondraček (02), $p^D(t,x,y) > 0$ when D is connected. Let D_1 and D_2 are two connected components of D, By Main Lemma that for any $x \in D_1$ and $B(x_0,r)$ with $\overline{B(x_0,r)} \subset D_2$, there exist $t_0 > 0$ and $c > 0$ such that

$$\begin{aligned}\mathbb{P}_x(X_t \in B(x_0,r), t < \tau_D) &\geq c \int_{D \setminus B(x_0,r)} G_D(x,y)dy \\ &\geq c \int_{D_1} G_{D_1}(x,y)dy > 0\end{aligned}$$

whenever $t \le t_0$. This implies that, for $t \le t_0$ and $x \in D_1$, $p^D(t,x,\cdot) > 0$ a. e. on D_2. By working with the dual process we get that for $t \le t_0$ and $y \in D_2$, $p^D(t,\cdot,y) > 0$ a. e. on D_1. Combining these with the semigroup property we get that $p^D(t,x,y) > 0$ on $(0,\infty) \times D_1 \times D_2$.

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Non-symmetric truncated processes

The process Y is said to be (non-symmetric) truncated process if the Lévy density $g(x)$ for Y is

$$g(x) := f(x)1_{\{|x|<1\}}$$

where f is the Lévy density for strictly α-stable process.

Roughly connected open set

Definition

D is roughly connected if $\forall x, y \in D$, there exist finite distinct connected components $U_1, \cdots, U_m$ of D such that $x \in U_1$, $y \in U_m$ and $\text{dist}(U_k, U_{k+1}) < 1$ for $1 \leq k \leq m-1$.

IU is true for killed non-symmetric truncated processes if D is a bounded, roughly connected κ-fat open sets.

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Idea

Let $h(x) := f(x) - g(x) = f(x)1_{\{|x| \geq 1\}}$. Since $\int_{\mathbb{R}^d} h(x)dx < \infty$, we can write $X_t = Y_t + Z_t$ where Z_t is a compound Poisson process with the Lévy density $h(x)$, independent of Y_t.

Let $T := \inf\{t \geq 0 : Z_t \neq 0\}$, an exponential random variable with intensity λ.

$Y_t = X_t$ for $t < T$ and $\{t < \tau_D^Y, t < T\} = \{t < \tau_D^X, t < T\}$ where $\tau_D^X := \inf\{t > 0 : X_t \notin D\}$ and $\tau_D^Y := \inf\{t > 0 : Y_t \notin D\}$.

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\leq \;& \mathbb{P}(X_t^D \in U, t < \tau_D^X \,|\, X_0 = x) \quad (\text{Equality if } \operatorname{diam}(D) < 1).
\end{aligned}
$$

Mixture

Suppose that X is a strictly α-stable process, Y is a truncated stable process and W is a Brownian motion in $\mathbb{R}^d$. We assume they are independent.

Then $W_t + X_t$ and $W_t + Y_t$ satisfy our assumptions.

- IU is true for any killed $W_t + X_t$ if D is bounded.
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Thank you!