SLE and α-SLE driven by Lévy processes

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Based on a joint work with Matthias Winkel.

Outline

1. Introduction of SLE

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2. SLE driven by Lévy processes

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3. α-SLE

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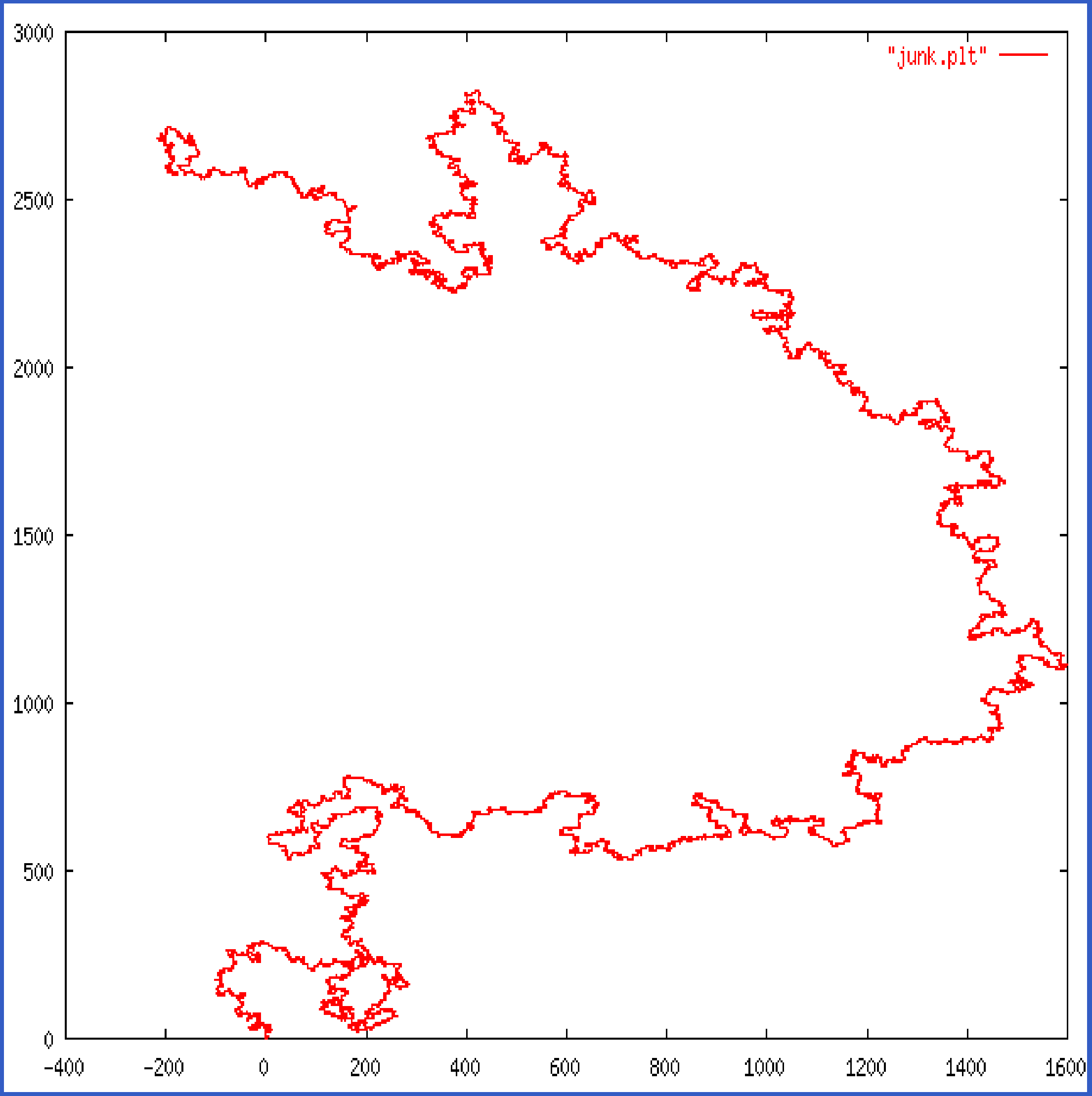
3. α-SLE
4. Further problems

1. Introduction of SLE

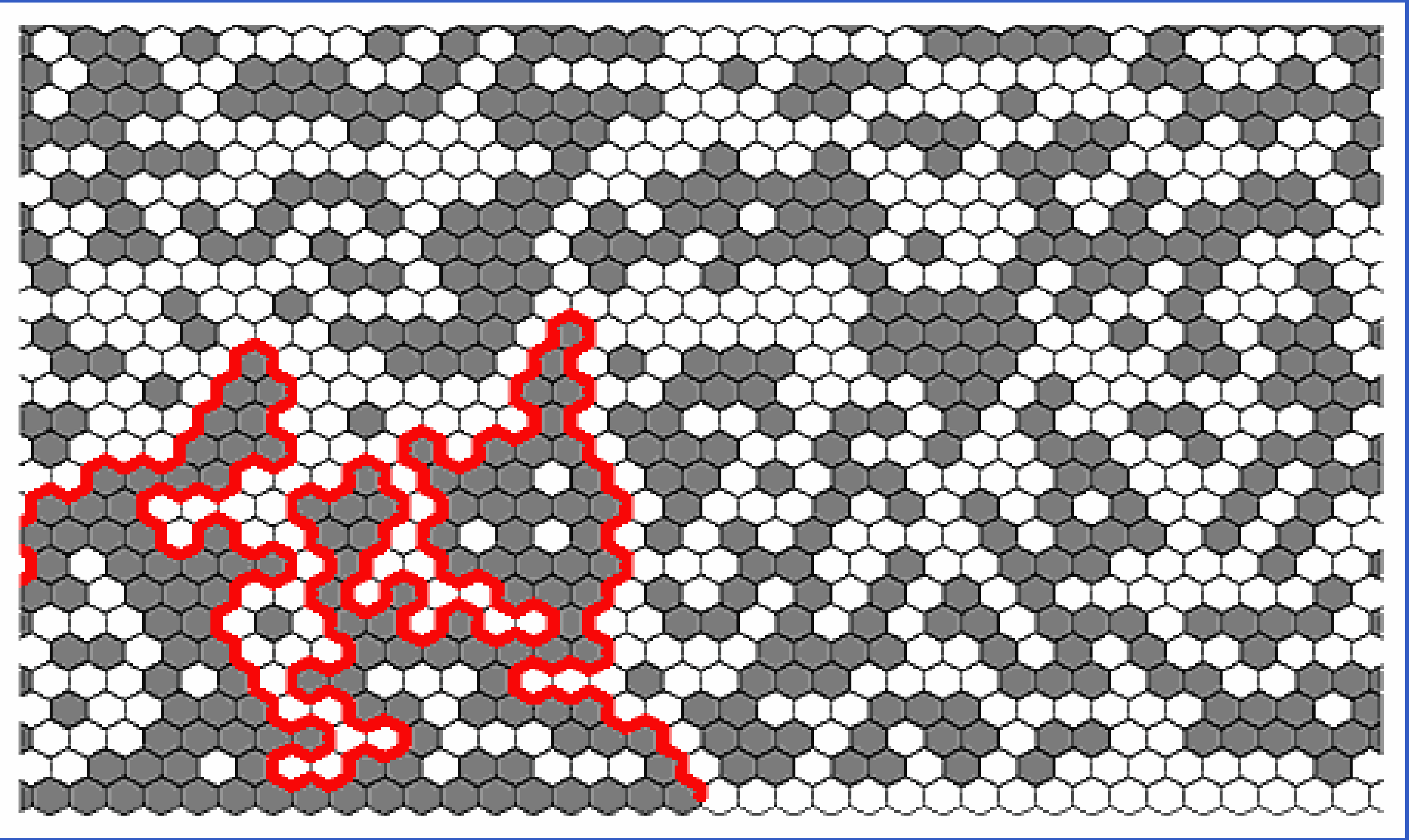
SLE: Stochastic(Schramm) Loewner Evolution

A new way to understand conformal invariant random increasing curves(sets) in complex plane.

Self-avoiding random walk, from Google



Interface of Ising model, from Google



Increasing curve and conformal map

Study curve by conformal map

Increasing curve and conformal map

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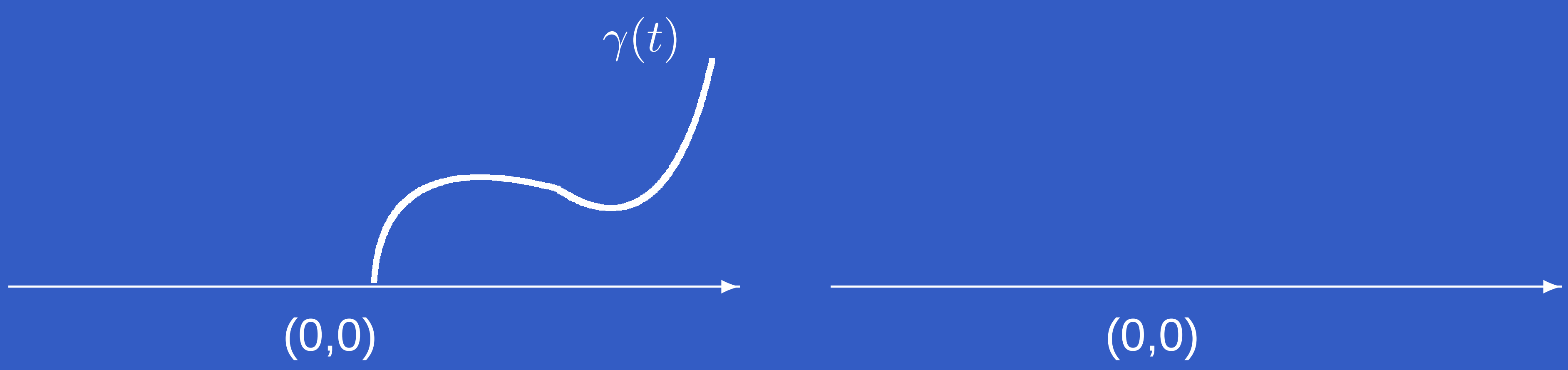

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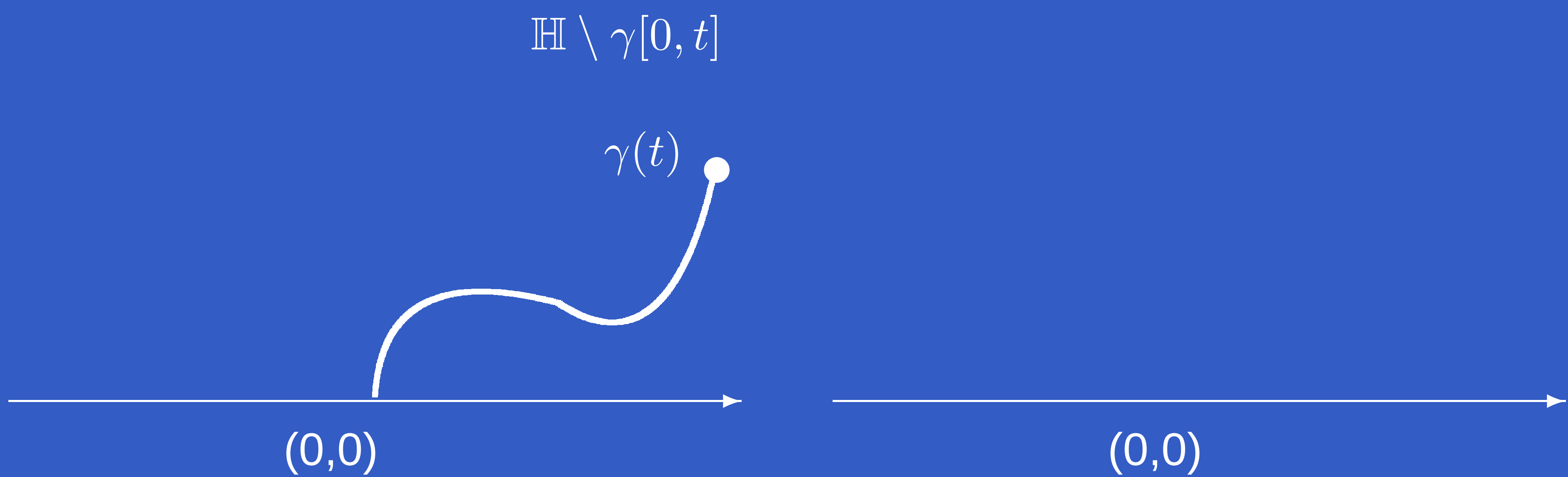

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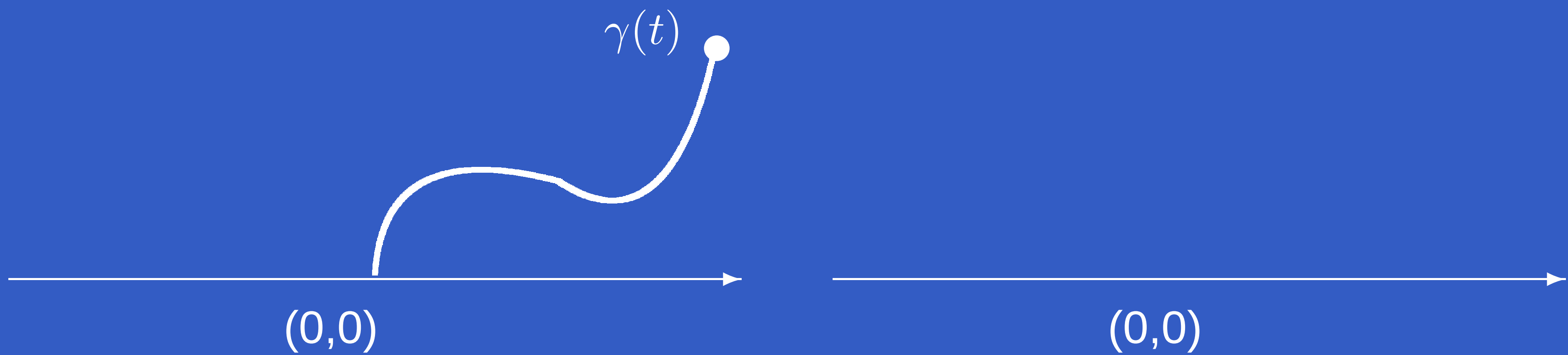
$$\mathbb{H} \setminus \gamma[0, t] \xrightarrow{g_t} \mathbb{H}$$



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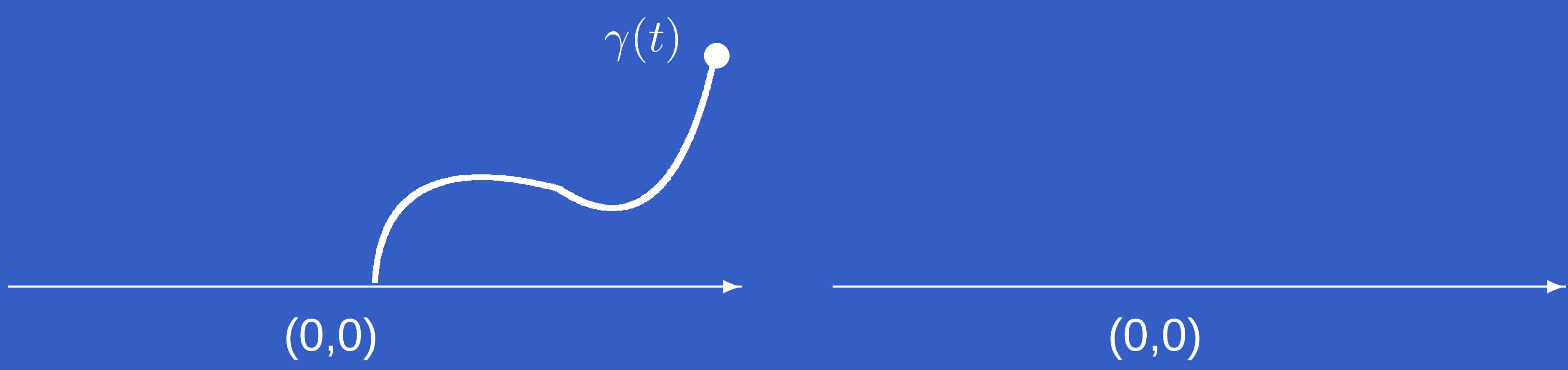

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$$\lim_{|z|\to\infty}(g_t(z) - z) = 0.$$

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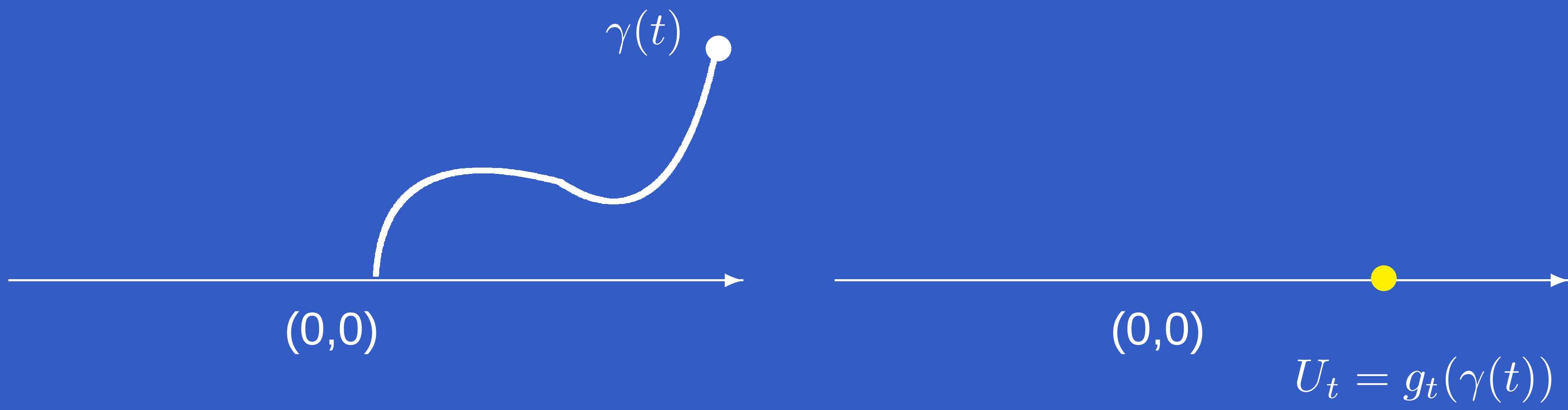

g_t is a unique conformal map if

$$\lim_{|z| \to \infty} (g_t(z) - z) = 0.$$

Loewner equation

Theorem If $(\gamma(t))_{t\geq 0}$ is a simple curve on $\mathbb{H}$, then $(g_t)_{t\geq 0}$ satisfies the following Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z, \qquad z \in \overline{\mathbb{H}},$$

where $U_t = g_t(\gamma(t))$ is a continuous real valued function.

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- For $z \in \overline{\mathbb{H}}$, function $g_{\cdot}(z)$ above can be solved on $[0, \zeta(z))$. Here

$$\zeta(z) = \sup\{t : |g_s(z) - U_s| > 0, \quad s \in [0, t]\}$$

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where $U_t = g_t(\gamma(t))$ is a continuous real valued function.

- For each $t \geq 0$, K_t is defined by the set of 'broken points' before time t, i.e.

$$K_t = \{z \in \overline{\mathbb{H}} : \zeta(z) \leq t\}.$$

Breakthrough

- In 1999, O. Schramm observed that if $(\gamma_t)_{t\geq 0}$ is conformal invariant(Markovian), then $(U_t)_{t\geq 0}$ is Brownian motion.

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- Similarly to the determined case, we can define stochastic increasing sets $(K_t)_{t\geq 0}$:

$$K_t = \{z \in \overline{\mathbb{H}} : \zeta(z) \leq t\}.$$

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  where $(B_t)_{t\geq 0}$ is a Brownian motion.

- $(K_t)_{t\geq 0}$ is called Schramm Loewner Evolution.

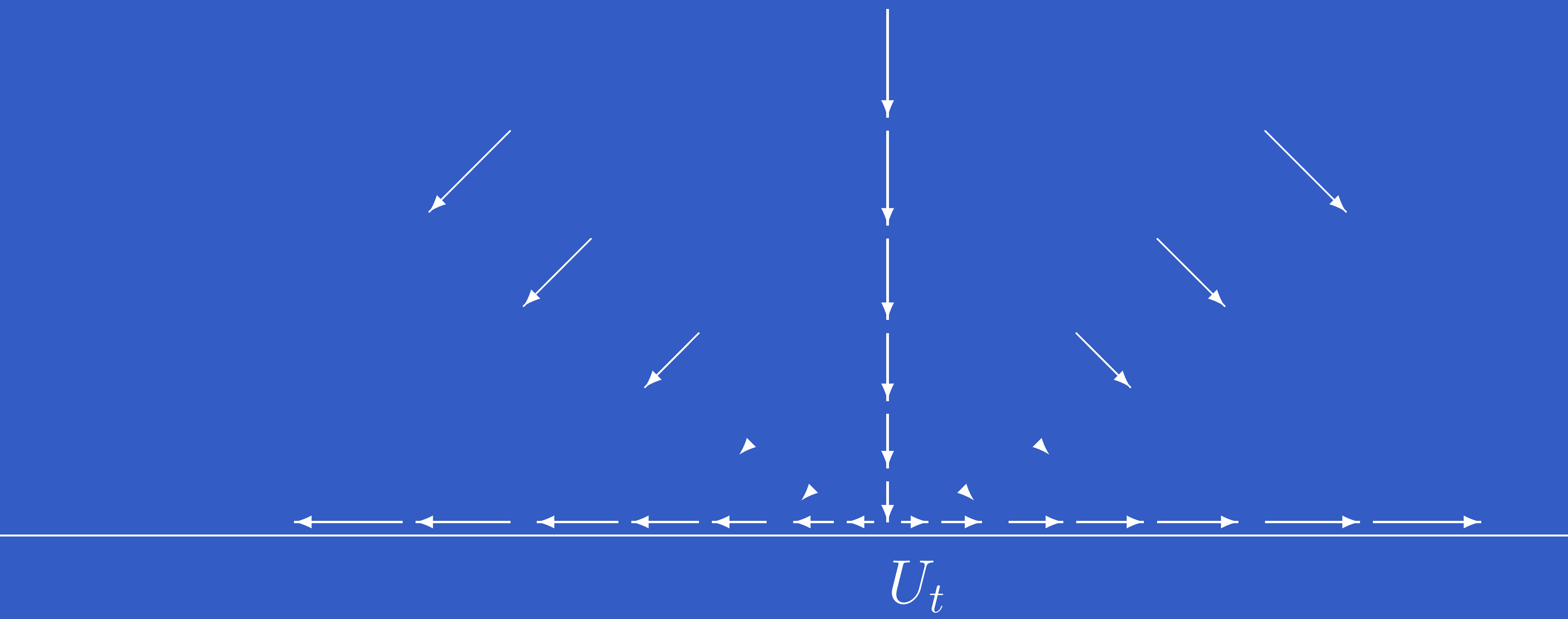
2. SLE driven by Lévy processes

- In what follows we consider driven processes

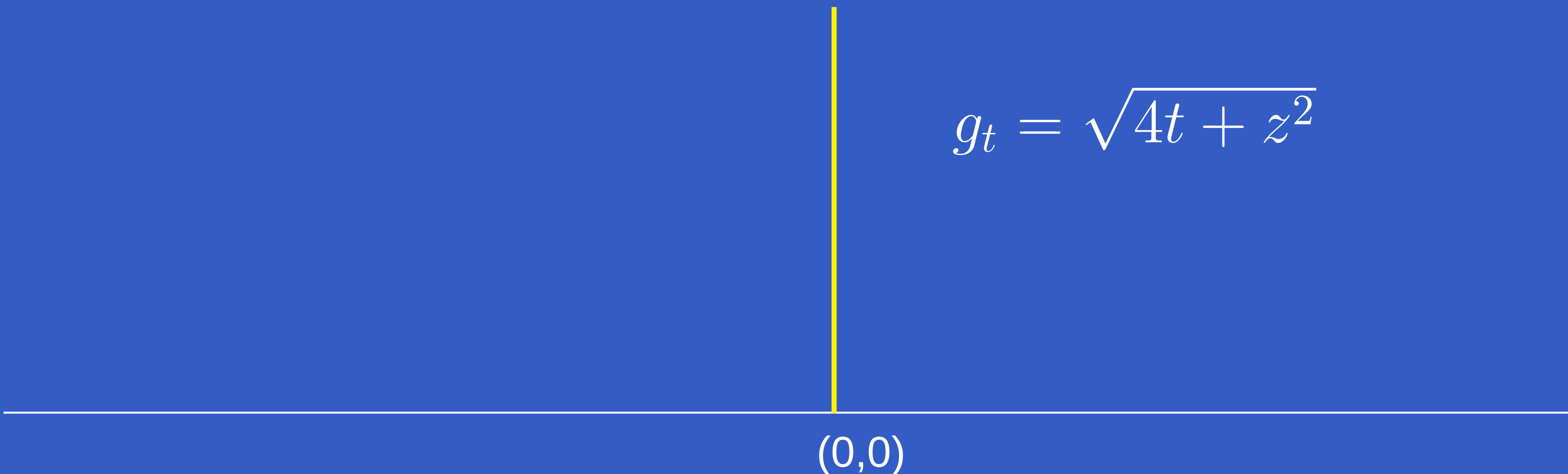
$$U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t,$$

where (B_t) and (S_t) are Brownian motion and symmetric α-stable process, respectively.

Vector fields of Stochastic Loewner equation



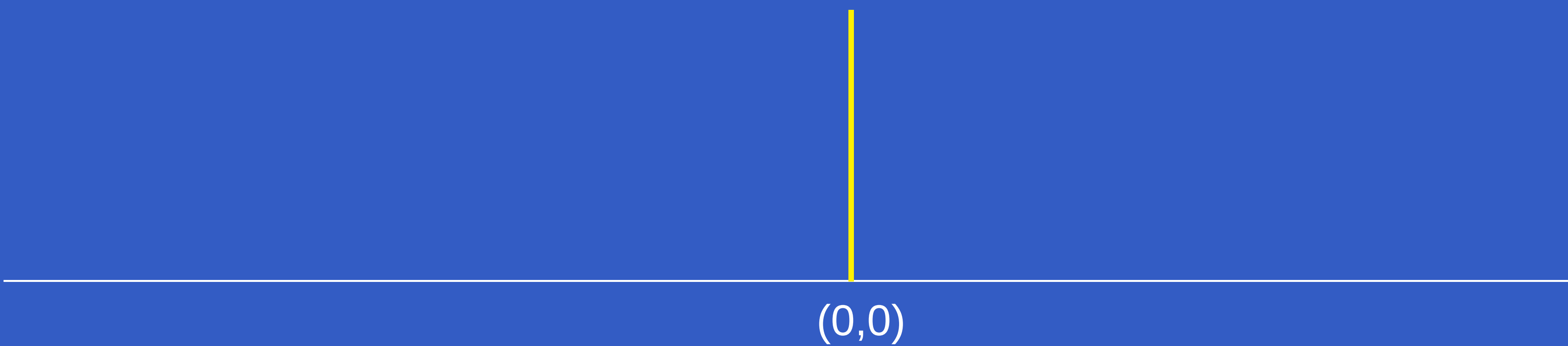
Example of continuous driven function



$$g_t = \sqrt{4t + z^2}$$

$$U_t \equiv 0, \; t \geq 0$$

Example of cádlág driven function



$$U_t = 0,\ t \in (0,1); \quad U_t = 1,\ t \in [1, \infty)$$

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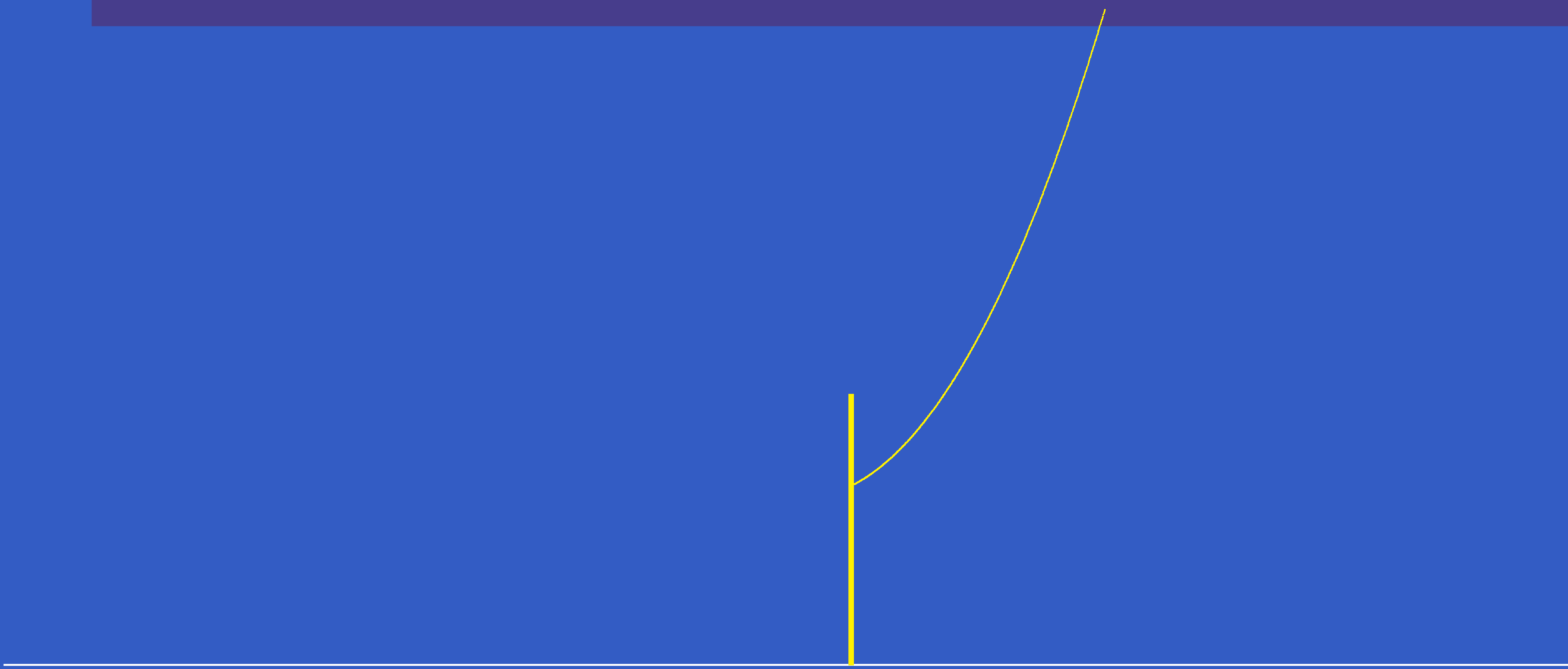

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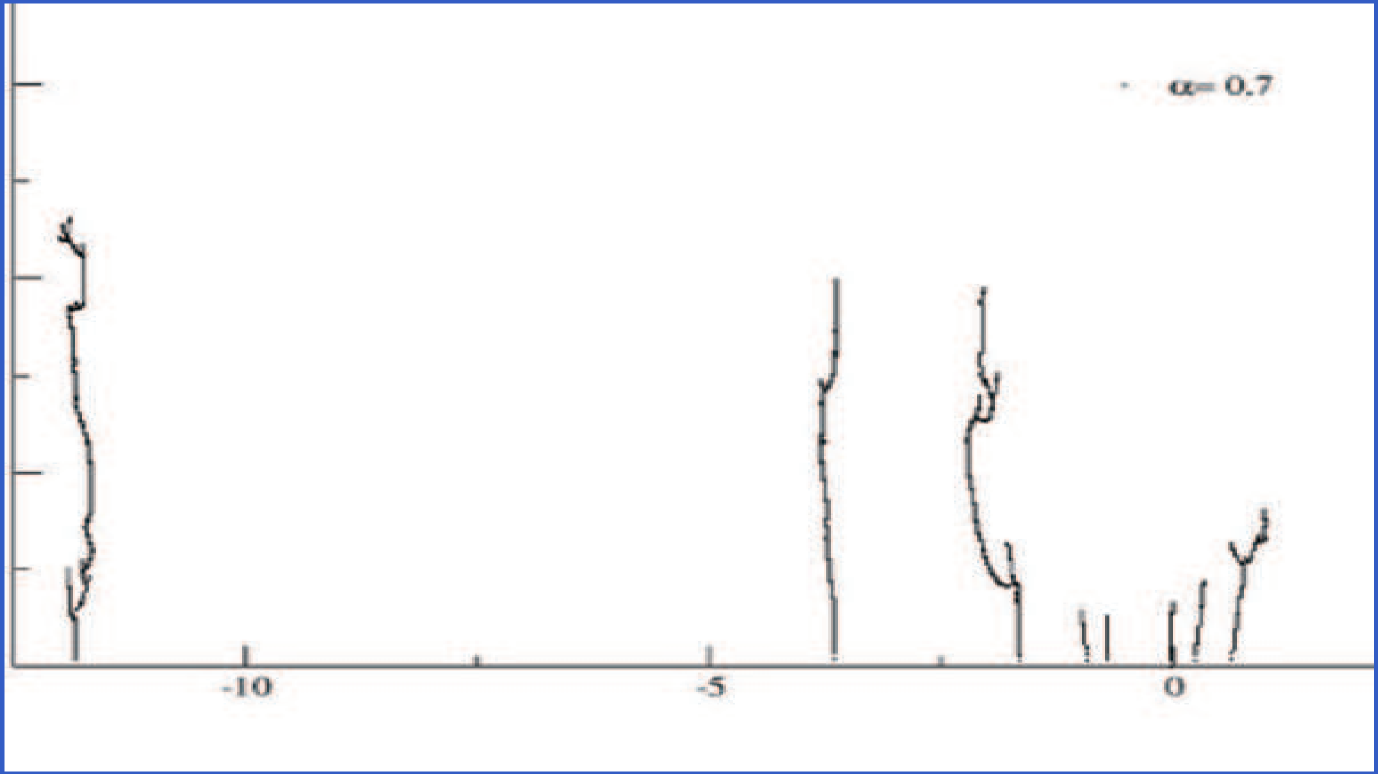
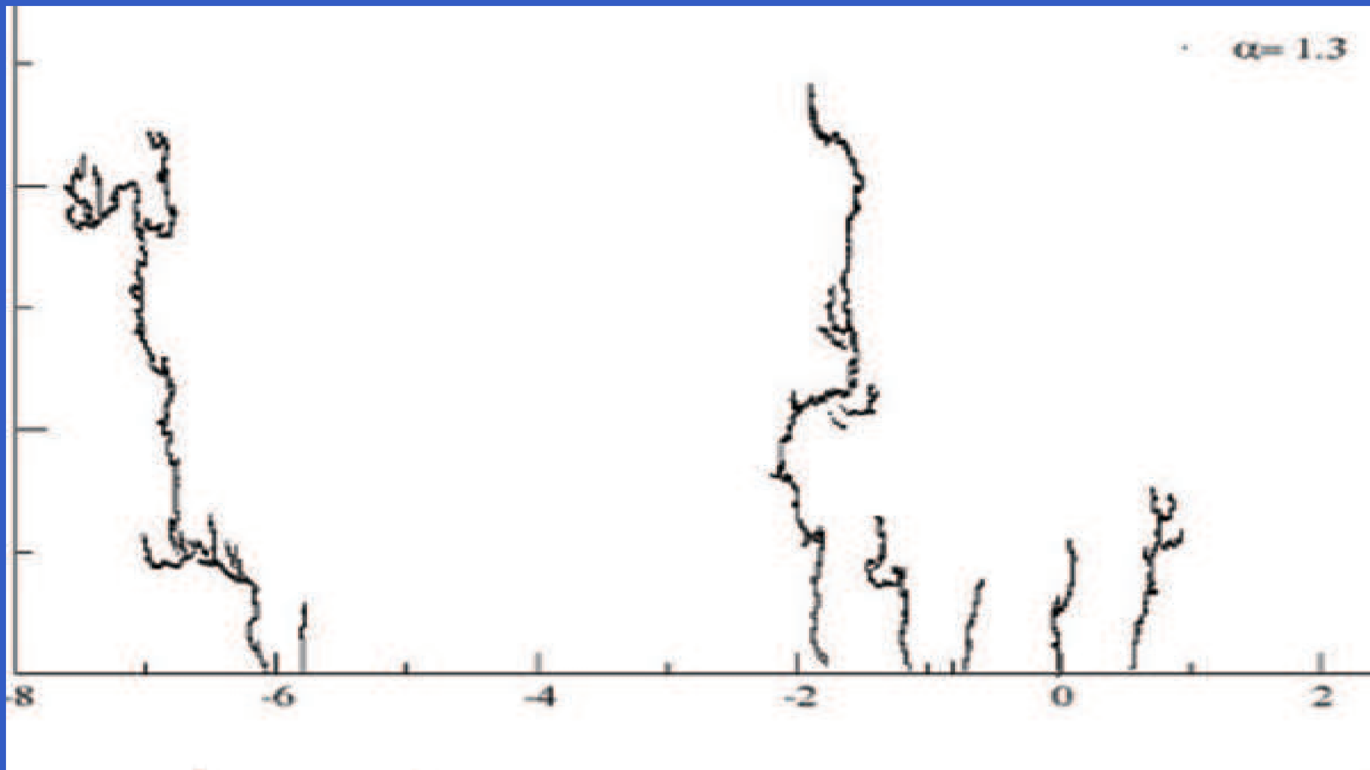
Simulation in [ROKG]



Take from I. Rushkin, P. Oikonomou, L.P. Kadanoff and I.A.Gruzberg, Stochastic Loewner evolution driven by Levy processes.

Simulation in [ROKG]



Phase transition

Theorem [GW,2006]
For each $z \in \overline{\mathbb{H}}$, denote the lifetime of the stochastic Loewner equation by $\zeta(z) = \inf\{t \geq 0 : z \in K_t\}$. Then

(i) if $0 \leq \kappa \leq 4$ and $U \not\equiv 0$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) = \infty) = 1$;

(ii) if $\kappa > 4$ and $1 \leq \alpha < 2$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) < \infty) = 1$;

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(iii) if $\kappa > 4$ and $0 < \alpha < 1$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $0 < \mathbb{P}(\zeta(z) < \infty) < 1$ and $\lim_{z \to 0, z \in \overline{\mathbb{H}} \setminus \{0\}} \mathbb{P}(\zeta(z) < \infty) = 1$.

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- Similarly to the Loewner equation, for each $t \geq 0$, we can define compact set K_t.
- If we take $U_t = \theta^{1/\alpha} S_t$, then

$$(K_{at})_{t\geq 0} = (a^{1/\alpha} K_t)_{t\geq 0}, \quad \text{in distribution.}$$

In this case we call $(K_t)_{t\geq 0}$ the α-SLE

Phase transition

Theorem [GW,2006] Let $1 < \alpha < 2$ and $(K_t)_{t\geq 0}$ the α-SLE driven by $U_t = \theta^{1/\alpha} S_t$ for a symmetric α-stable process S. Set

$$\theta_0(\alpha) = 2/(\mathcal{A}(1,-\alpha)|\gamma(\alpha,1)|).$$

(i) if $0 < \theta < \theta_0(\alpha)$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) = \infty) = 1$;

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Transform

- Let $h_t(z) = g_t(z) - \theta^{1/\alpha} S_t$, then we have

$$dh_t(z) = \frac{2|h_t(z)|^{2-\alpha}}{h_t(z)} dt - \theta^{1/\alpha} dS_t, \qquad h_0(z) = z, \quad z \in \overline{\mathbb{H}} \setminus \{0\}.$$

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- When $x \in \mathbb{R}$, $(h_t(x))_{t\geq 0}$ is an $\mathbb{R}$-valued Markov process and its generator $A^{\alpha,\theta}$ acting on C^2 function f is

$$A^{\alpha,\theta} f(y) = \frac{|y|^{2-\alpha}}{y} \partial_y f(y) + \theta \Delta_y^{\alpha/2} f(y), \qquad \text{for all } y \neq 0.$$

Harmonic function

- Lemma[GW,2006] For $p \in \mathbb{R}$, define a function $w_p : \mathbb{R} \to \mathbb{R}$ by $w_p(0) = 0$ and

$$w_p(x) = |x|^{p-1}, \ \ x \in \mathbb{R} \setminus \{0\}, p \neq 1; \quad w_1(x) = \ln|x|, \ \ x \in \mathbb{R} \setminus \{0\}.$$

Then,

$$\Delta_x^{\alpha/2} w_p(x) = \mathcal{A}(1, -\alpha)\gamma(\alpha, p)|x|^{p-\alpha-1}, \ \ x \in \mathbb{R} \setminus \{0\}, \ p \in (0, \alpha+1),$$

where $\gamma(\alpha, p) = \alpha^{-1}(p-1) \int_0^\infty v^{p-2}(|v-1|^{\alpha-p} - (v+1)^{\alpha-p})\, dv$ for $p \neq 1$ and $\gamma(\alpha, 1) = \alpha^{-1} \int_0^\infty v^{-1}(|v-1|^{\alpha-1} - (v+1)^{\alpha-1})\, dv$.

4. Further Problems

Random càdlàg curves

- **Theorem** [G1,2007]
  Let $0 < \alpha < 2$, $\theta \geq 0$, $\kappa \in [0,4) \cup (4,8) \cup (8,\infty)$ and $f_t = g_t^{-1}$. Then, almost surely, the conformal maps $(f_t)_{t \geq 0}$ extend to $\overline{\mathbb{H}}$ continuously and $(K_t)_{t \geq 0}$ is generated by right continuous curve with left limit.

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- Are these curves transient.

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- How about the high dimensional cases.

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Thank you!