Random Continued Fractions

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Continued Fractions : Quick Review

Given a non-negative integer a_0 and a finite sequence of positive integers $a_0, a_1, \ldots, a_n$, consider the expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

This is called a Continued Fraction. Standard Notation is $$[a_0; a_1, \ldots, a_n].$$

What if we have an infinite sequence $a_1, a_2, \ldots$ of positive integers? Can one define a Continued Fraction? How?

Natural course is to define $x_n = [a_0 : a_1, \ldots, a_n]$, for each $n \geq 1$, and then ask if the sequence x_n converges.

There is an elegant way of doing this. Given any finite or nonterminating sequence $a_0, a_1, \dots$ of integers with $a_0 \geq 0$ and $a_i > 0,\ i \geq 1$, one constructs two sequences $p_0, p_1, \dots$ and $q_0, q_1, \dots$ as follows:

$$p_0 = a_0,\ q_0 = 1; \qquad p_1 = a_0 a_1 + 1,\ q_1 = a_1;$$

and

$$\text{for } k \geq 2,\ \ p_k = a_k p_{k-1} + p_{k-2},\ q_k = a_k q_{k-1} + q_{k-2}.$$

It is then easy to see that $x_k = [a_0; a_1, \dots, a_k] = \dfrac{p_k}{q_k}$. It

is also easy to see that

$x_0 < x_2 < \cdots \cdots < x_3 < x_1$, that is, $\{x_{2k}\} \uparrow,\ \{x_{2k+1}\} \downarrow$.

In case the sequence $a_0, a_1, \dots$ is non-terminating, convergence of the sequence $\{x_n\}$ follows from

- $p_n \geq p_{n-2},\ q_n \geq q_{n-2}$; in particular, $q_{2n} \geq 1$ and $q_{2n+1} \geq a_1 \geq 1$ for all n.

- $q_n q_{n-1} = a_n q_{n-1}^2 + q_{n-1} q_{n-2} = \cdots$
  $= a_n q_{n-1}^2 + a_{n-1} q_{n-2}^2 + \cdots + a_1 q_0^2 \geq \sum_2^n a_k \uparrow \infty.$

- $$x_n - x_{n-1} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}} = -\frac{p_{n-1} q_{n-2} - p_{n-2} q_{n-1}}{q_n q_{n-1}}$$
  $$= \cdots = (-1)^n \frac{p_1 q_0 - p_0 q_1}{q_n q_{n-1}} \quad \to \quad 0.$$

The limit $x = \lim x_n$ is referred to as the infinte (non-terminating) continued fraction denoted by

$$x = [a_0; a_1, \ldots]$$

and the x_n as defined above is called the nth convergent of x.

When $a_0 = 0$, one writes $[a_1, a_2, \ldots]$ for $[0; a_1, a_2, \ldots]$ in both the terminating as well as non-terminating cases. It is clear that terminating continued fractions are rationals. In particular, the convergents x_n for an infinte contiunued fraction x (which is always an irrational) give a sequence of rationals converging to x.

Fact: Any positive real x can be expressed as a (terminating or non-terminating) continued fraction.

Clearly $a_0 = [x]$. So it is enough to do this for $x \in (0, 1)$. It is done through the **Gauss map** defined as

$$T : [0, 1) \longrightarrow [0, 1),\ T0 = 0 \text{ and, for } 0 < x < 1,\ Tx = \frac{1}{x} - \left[\frac{1}{x}\right].$$

For any $x \in (0, 1)$, consider the orbit of x:

$$y_0 = x, y_1 = Ty_0 = Tx, y_2 = Ty_1 = T^2x, \ldots, \text{ and so on.}$$

Two possibilities:

- There is $n \geq 1$ such that $y_n = 0$ but $y_0, \ldots, y_{n-1}$ are all > 0. Taking $a_k = [\frac{1}{y_{k-1}}]$, $k = 1, \ldots, n$, one gets $x = [a_1, \ldots, a_n]$.

- $y_n > 0$ for all n. Taking the non-terminating positive integer sequence $a_k = [\frac{1}{y_{k-1}}]$, one gets $x = [a_1, a_2, \ldots]$.

Thus the above algorithm produces, for every $x \in (0,1)$, a (finite or infinite) positive integer sequence $a_1, a_2, \ldots$ such that $x = [a_1, a_2, \ldots]$. Constructing (p_k, q_k) and the kth convergent x_k as before one gets

$$x_2 \leq x_4 \leq \cdots \leq x \leq \cdots \leq x_3 \leq x_1 \text{ and } x_k \to x.$$

Well-known facts:

- Terminating sequence $\Leftrightarrow$ Finite continued Fraction $\Leftrightarrow$ 0 is in the orbit $\Leftrightarrow$ x rational.

- Periodic or eventually periodic orbit $\Leftrightarrow$ x quadratic irrational. [Ex: $[1, 1, \ldots] = (\sqrt{5}-1)/2$ (Golden number).

- [Gauss - 1812]: $m_n = \lambda(T^n)^{-1}$ where λ is lebesgue measure on $[0,1)$. For $0 < y < 1$, (Kuzmin-1928)

  $$m_n[(0,y]] = \lambda\{x : T^n x \leq y\} \longrightarrow \frac{\log(1+y)}{\log 2}.$$

  $$d\mu(y) = \frac{1}{\log 2}\frac{1}{1+y}dy, \quad 0 < y < 1,$$

  is the unique (abs conts) invariant measure for the dynamical system. The system is ergodic and ϕ-mixing.

- Khinchin – Levy – Doeblin – Billingsley :

  $$\frac{a_1 + \cdots + a_n}{n} \to \infty, \text{ and, } \frac{\log q_n}{n} \to \pi^2/(12\log 2), \text{ a.e.}$$

In defining the continued fractions $[a_0; a_1, a_2, \ldots]$, one can take **non-integer inputs** as well. p_k, q_k and x_k will still be well-defined, as long as $a_0 \geq 0$ and $a_k > 0, k \geq 1$. For convergence in the non-terminating case, a sufficient condition now will be that $\sum a_k$ **diverges**.

For defining infinite continued fractions, it is not even necessary to start with $a_k > 0$ for all $k \geq 1$.

Just $a_{2k+1} > 0$ for **some** $k \geq 1$ will ensure $q_k > 0$ and hence $x_k = p_k/q_k$ is well-defined for **all large** k. For convergence of $\{x_k\}$, divergence of $\sum a_k$ remains sufficient.

Suppose now that $a_0, a_1, a_2, \ldots$ is a **randomly chosen** non-negative sequence. $[a_0; a_1, a_2, \ldots]$ may or may not be well-defined. If well-defined, then it is a random variable; what kind of random variable? We consider the special case: **I.I.D. Sequence**

- Existence of $[a_0; a_1, a_2, \ldots]$ is trivially settled.
- Some general results on distribution of $[a_0; a_1, a_2, \ldots]$
- Interesting interplay with **Markov chain theory, Invariant distribution**
- Explicit expression obtained for the distribution in a special case (interesting distribution!)
- Extensions to higher dimensions.

Random Continued Fractions

$Z_0, Z_1, Z_2, \ldots$ are **I.I.D. Non-Negative** random variables. Define random variables p_k, q_k, $k \geq 0$ recursively:

$$p_0 = Z_0,\ q_0 = 1; \qquad p_1 = Z_0 Z_1 + 1,\ q_1 = Z_1; \text{ and,}$$
$$p_k = p_{k-1} Z_k + p_{k-2}, \quad q_k = q_{k-1} Z_k + q_{k-2}, \quad k \geq 2$$

Assume $P(Z_i = 0) < 1$.

Theorem (Bhattacharya & G): With probability one, $p_k / q_k = [Z_0; Z_1, \ldots, Z_k]$ is well-defined for all large k and the limit $Y = \lim_{k \to \infty} [Z_0; Z_1, \ldots, Z_k]$ exists.

It is clear that the limit random variable Y is **strictly positive**. Thus its distribution is a probability on $(0, \infty)$. What can we say about this probability? In particular, given a specific I.I.D. sequence $\{Z_i,\ i \geq 0\}$, can we find the distribution of $Y = [Z_0;\ Z_1,\ Z_2, \ldots]$?

A Related Markov Chain:

Consider the Markov Chain $\{X_n,\ n \geq 0\}$ on the State Space $S = (0, \infty)$ defined as follows:

X_0 strictly positive random variable independent of $\{Z_i\}$;

$$X_1 = Z_1 + \frac{1}{X_0} = [Z_1; X_0], \quad X_2 = Z_2 + \frac{1}{X_1} = [Z_2; Z_1, X_0];$$

And in general, $X_n = Z_n + \frac{1}{X_{n-1}} = [Z_n; Z_{n-1}, \ldots, Z_1, X_0]$.

$S = (0, \infty)$ $\quad f_z(x) = z + \frac{1}{x}$ $\quad f_z : (0, \infty) \to (0, \infty)$

If the chain has a **Unique Invariant Distribution** π on $S = (0, \infty)$ and if $\{X_n\}$ **converges weakly** to π, then π is characterized by the distributional identity

$$X \stackrel{\mathcal{L}}{=} Z + \frac{1}{X}$$

$$(Z \stackrel{\mathcal{L}}{=} Z_0; \quad X \stackrel{\mathcal{L}}{=} \pi; \quad X \text{ independent of } Z)$$

Assume $\boxed{Z_i \text{ are non-degenerate}}$

Denote μ to be the initial distribution (distribution of X_0) and μ_n to be the distribution of X_n.

<u>**Theorem**</u> (Bhattacharya & G):

(a) The chain $\{X_n\}$ has a **unique invariant distribution** π that doesn't depend on the initial distribution μ.

(b) μ_n **converges to** π **in Kolmogorov distance** exponentially fast uniformly over μ.

(c) The invariant distribution is **non-atomic**.

The proof is based an extension (due to Bhattacharya & G) of an old idea of Dubins-Freedman(1966).

The idea: Consider Markov chains that arise as **Random Iterations of Functions (RIF)**.

- S the state space; $\mathcal{H}$ a class of functions $S \to S$; μ a probability on $\mathcal{H}$.
- Start with any initial S-valued random variable X_0 and define X_n by

  $$X_n = f_n(X_{n-1}),\ n \geq 1$$

  where f_n are IID with distribution μ.

Diaconis-Freedman(2000) gives an excellent survey on such Markov chains (RIFs). Dubins-Freedman considered the special case when S is a compact interval in $\mathbf{R}$ and $\mathcal{H}$ is a class of monotone increasing functions. They introduced what has since then been known as the **Splitting Condition** and showed that this condition is sufficint to prove existence of a unique invariant probability and also exponential convergence in Kolmogorov distance. We needed to extend this to non-compact intervals and also when $\mathcal{H}$ consists of monotone functions. This turned out to be somewhat non-trivial.

Non-atomicity of π is a simple consequence of the uniqueness of the invariant distribution, the convolution identity characterizing π and the convergence of μ_n to π in Kolkmogorov distance

$$d(\mu_n, \pi) = \sup_x |F_{\mu_n}(x) - F_\pi(x)|.$$

What is the connection of this invariant probability π with our infinite random continued fraction

$$Y = [Z_0; Z_1, Z_2, \ldots] = \lim_n [Z_0; Z_1, \ldots, Z_n]?$$

Defining

$$Y_n = [Z_0; Z_1, \ldots, Z_{n-1}, X_0],$$

it is clear that

$$Y_n \stackrel{\mathcal{L}}{=} X_n \text{ for each } n,$$

so that

$$Y_n \text{ converges in distribution to } \pi.$$

A little work (using estimates from the classical theory of continued fractions) shows that

$$\boxed{Y_n \longrightarrow Y \text{ with probability one,}}$$

Thus one gets

$$\boxed{Y \stackrel{\mathcal{L}}{=} \pi}$$

Thus we have our first conclusion about the distribution of the random infinite continued fraction

$$Y = [Z_0; Z_1, Z_2, \ldots]$$

when $Z_i \geq 0$, $i \geq 0$ are IID with $P(Z_i = 0) < 1$.

- Z_i **non-degenerate** $\Longrightarrow$ Y **has a non-atomic distribution.**

Theorem (Bhattacharya & G): Suppose that the common distribution of the Z_i satisfies the conditions

$$P\,[0 < Z_i \leq 1] > 0,$$

and

$$P\,[Z_i < \epsilon] > 0, \quad \text{for all } \epsilon > 0.$$

Then the distribution π of Y has **Full Support**, i.e., **support** $(\pi) = (0, \infty)$.

The proof rests on the following two lemmas; the first one closely follows classical arguments on representation of all positive reals by continued fractions, while the second one may be of independent interest in general Markov chain theory.

Lemma 1 (Bhattacharya & G): For $0 < \theta \leq 1$, the set of all continued fractions $[a_0\theta; a_1\theta, a_2\theta, \ldots]$ where $\{a_0, a_1, a_2, \ldots\}$ is a terminating or non-terminating sequence of integers with $a_0 \geq 0$ and $a_i > 0 \; \forall \, i \geq 1$, comprises all of $(0, \infty)$.

Lemma 2 (Bhattacharya & G): Let $p^{(n)}(x, dy)$ be the n-step transition probability of a Markov Chain on $(S, \mathcal{S})$, where S is a separable metric space and $\mathcal{S}$ its Borel σ-field, and let π be an invariant probability. Assume that the map $x \mapsto p^{(1)}(x, dy)$ is weakly continuous on S. If x_0 belongs to the support of π and if $p^{(n)}(x_0, dy)$ converges weakly to π, then the support of π coincides with the closure of $\bigcup_{n=1}^{\infty} S_n(x_0)$ where $S_n(x_0)$, for each n, denotes the support of $p^{(n)}(x_0, dy)$.

Some Special Cases:

Ex 1: Gamma Innovation

<u>**Theorem**</u> (Letac & Shesadri): If the common distribution of the $\{Z_i\}$ is a Gamma distribution with parameters λ and a, then the distribution π of Y is given by the density

$$g_{\lambda,a}(x) = (2K_\lambda(2a))^{-1}x^{\lambda-1}e^{-a(x+\frac{1}{x})}\ , \quad x \in (0,\infty),$$

where $K_\lambda(\cdot)$ denotes the Bessel function.

The proof is based on use of Laplace transforms. In that sense it is lucky case!

The above density is a special case of what is known as **generalized inverse gaussian distribution** [Barndorff-Nielsen and Halgreen (1977)], given by density

$$g_{\lambda,a,b}(x) = \frac{a^{\lambda/2}b^{-\lambda/2}}{2K_\lambda(\sqrt{ab})}\, x^{\lambda-1}\exp\{\frac{1}{2}(ax+bx^{-1})\}\ , \quad x \in (0,\infty),$$

with parameters $\lambda \neq 0$ and $a > 0,\ b > 0$.

Easy to see:

$$X \text{ has density } g_{\lambda,a,b} \Longrightarrow \frac{1}{X} \text{ has density } g_{-\lambda,b,a}.$$

The theorem says that the continued fraction with IID Gamma (λ, a) innovations has generalized inverse gaussian distribution with parameters λ, $2a$ and $2a$.

Ex 2: Bernoulii Innovation

$$P(Z_i = 0) = \alpha, \;\; P(Z_i = 1) = 1 - \alpha. \qquad (0 < \alpha < 1)$$

<u>**Theorem**</u> (Bhattacharya & G; also, Chassaing et al.):

(a) $Y = [Z_0; Z_1, Z_2, \ldots]$ has distribution function F given by

$$\begin{aligned} F(x) &= \sum_{i \geq 0} \left(-\frac{1}{\alpha}\right)^i \left(\frac{\alpha}{1+\alpha}\right)^{a_1 + \cdots + a_{i+1}}, & 0 < x \leq 1, \\ F(x) &= 1 - \frac{1}{\alpha} F\left(\frac{1}{x}\right), & x > 1, \end{aligned}$$

where $x = [0; a_1, a_2, \ldots]$ is the usual continued fraction expansion of $x \in (0, 1]$, with the the sum on the right of the first equation being a finite sum in case the continued fraction expansion $[0; a_1, a_2, \ldots]$ of x is terminating;

(b) the distribution F is **Singular with Full Support** $S = (0, \infty)$.

Getting the explicit expression for F as in (a) is a matter of playing around with the convolution identity characterizing π.

The proof of singularity in (b) is an interesting one! One proves that

$F'(x) = 0$ for [Lebesgue] almost every irrational $x \in (0, 1)$.

Write $x = [0;\, a_1,\, a_2, \ldots]$ (non-terminating!).

Denote $x_n = [0;\, a_1,\, \ldots,\, a_n]$.

Since the right of the first relation in (a) is an alternating series with terms decreasing in magnitude,

$$\begin{aligned}|F(x) - F(x_n)| &= \left| \sum_{i=n}^{\infty} \left(-\frac{1}{\alpha}\right)^i \left(\frac{\alpha}{1+\alpha}\right)^{a_1+\cdots+a_{i+1}} \right| \\ &\leq \frac{1}{\alpha^n} \left(\frac{\alpha}{1+\alpha}\right)^{a_1+\cdots+a_{n+1}}\end{aligned}$$

Writing $x_n = \dfrac{p_n}{q_n}$, where p_n and q_n are relatively prime integers, and using classical inequalities,

$$|x - x_n| > \frac{1}{q_n(q_{n+1}+1)},$$

so that,

$$\left|\frac{F(x) - F(x_n)}{x - x_n}\right| < \frac{1}{\alpha^n}\left(\frac{\alpha}{1+\alpha}\right)^{a_1+\cdots+a_{n+1}} q_n(q_{n+1}+1).$$

Let M be chosen large enough to satisfy

$$\left(\frac{\alpha}{1+\alpha}\right)^M \cdot \frac{\gamma^2}{\alpha} < 1,$$

where $\gamma = \exp\{\pi^2/(12\log 2)\}$. It follows from classical Levy-Khinchin limit theorems that for all x outside a set of Lebesgue measure zero, one has

$$a_1 + \cdots + a_{n+1} > M(n+1) \quad \forall \text{ all sufficiently large } n.$$

It now follows immediately that for all x outside a set of Lebesgue measure zero,

$$\lim_{n\to\infty}\left|\frac{F(x) - F(x_n)}{x - x_n}\right| = 0.$$

Lebesgue Differentiation Theorem completes the job now!

Chassaing et al's argument:

Elements of $SL(2,\mathbf{R})$ (2×2 real matrices with determinant 1) define a projectivity on $\mathbf{R} \cup \{\infty\}$:

$$x \mapsto A(x) = \frac{ax+b}{cx+d}\,.$$

($A(-d/c) = \infty$, $A(\infty) = a/c$, if $c \neq 0$.) The idea used is to consider the Markov chain given by the products of IID random matrices applied to an initial value. The matrices are chosen with a special common distribution. The argumnets are a bit complicated and the proof of singularity is not direct (claimed to be "mere adoption of " certain arguments of S.D. Chatterji and F. Schweiger!). A more direct proof and additional information is obtained in

Theorem (Chakrabarti & Rao): For various $0 < \alpha < 1$, the distributions F_α are all singular with respect to Lebesgue measure and with respect to one another. Moreover, the family $\{F_\alpha\}$ is a uniformly singular family.

Consider a slight departure from standard Bernoulli!

$$\{Z_i\} \text{ IID },\quad P\left(Z_i = 0\right) = \alpha,\ P\left(Z_i = \theta\right) = 1 - \alpha$$

$$(\theta > 0, 0 < \alpha < 1)$$

Theorem (Bhattacharya & G):

(a) $\theta \leq 1 \Longrightarrow$ support of π is **Full**, namely, $(0,\infty)$.

(b) $\theta > 1 \Longrightarrow$ support of π is a **Cantor subset** of $(0,\infty)$.

Extension to Higher Dimensions

[Hallin]

$H_n^+(\mathbf{R})$: real $n \times n$ symmetric positive definite matrices. For $A_0, A_1, A_2, \ldots$ in $H_n^+(\mathbf{R})$, one defines

$$[A_0] = A_0, [A_0; A_1] = A_0 + A_1^{-1},$$
$$\text{and } [A_0; A_1, \ldots A_n] = A_0 + [A_1; A_2, \ldots, A_n]^{-1}, ; n \geq 2$$

These are the finite continued fractions. **Well-defined!**

$[A_1; A_2, \ldots, A_n]$ all **invertible**.

For convergence of the sequence $\{[A_0; A_1, \ldots, A_n]\}$, one uses arguments anlogous to the one-dimensional case, namely, that with matrices P_n and Q_n defined recursively by

$$P_0 = A_0, Q_0 = I, \ P_1 = A_1 A_0 + I, Q_1 = A_1$$
$$\text{and, for } n \geq 2, \ P_n = A_n P_{n-1} + P_{n-2}, Q_n = A_n Q_{n-1} + Q_{n-2}$$

one can show that the matrix $Q'_{n+1} Q_n$ is in $H_n^+(\mathbf{R})$ (in particular, invertible) and that

$$[A_0; A_1, \ldots, A_{n+1}] - [A_0; A_1, \ldots, A_n] = (-1)^{n+1} (Q'_{n+1} Q_n)^{-1}.$$

The convergence of the sequence $\{[A_0; A_1, \ldots, A_n]\}$ can now be proved by showing convergence of the alternating series $\sum_{n=0}^{\infty} (-1)^{n+1} (Q'_{n+1} Q_n)^{-1}$.

Given now any i.i.d. sequence $Z_1, Z_2, \ldots$ of random matrices taking values in $H_n^+(\mathbf{R})$, consider the Markov Chain on $S = H_n^+(\mathbf{R})$ defined recursively by

$$X_1 = [Z_1; X_0], \ X_2 = [Z_2; X_1],$$
$$\text{and, in general, } X_{n+1} = [Z_{n+1}; X_n], \text{ for } n \geq 1,$$

where X_0 is any $H_n^+(\mathbf{R})$-valued random matrix, independent of the sequence $\{Z_i\}$. Using arguments similar to the one-dimensional case, one can show that this markov chain has a unique invariant distribution π and that X_n converges in distribution to π, whatever be X_0. Moreover, this π is the unique probability on $H_n^+(\mathbf{R})$ satisfying the distributional identity

$$X \stackrel{\mathcal{L}}{=} Z + X^{-1},$$

where X and Z are independent $H_n^+(\mathbf{R})$-valued random matrices with X having distribution π and $Z \stackrel{\mathcal{L}}{=} Z_1$.
In fact, Bernadac has extended the notion of continued fractions on a more abstract space of irreducible symmetric cones, i.e., the interior of cones of square elements of simple Euclidean Jordan algebras. For Bernadac (1995) has more details.

Existence of a unique invariant probability and properties of this invariant probability all go through using appropriate extensions of the one-dimensional arguments.

THANK YOU!