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Essential spectral radius for positive operators on L^1 and L^∞ spaces

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1. Introduction

Let μ be a probability measure on a Polish space E with its Borel σ-algebra $\mathcal{B}$, and $\pi : L^p(E,\mu) \to L^p(E,\mu)$ be a positive operator with $1 \le p \le \infty$.

We introduced in [Gong, Wu: C.R. Acad. Sci. Pairs, t.331, Série 1(2000),[3]], [Gong, Wu: J. Math. Pures Appl. 85(2006),151-191, [4]], and [Wu: Theory Relat. Fields 128(2004), 255-321, *Remarks(5.ii)* in [12]] the conceptions:

tail norm of π and tail norm condition (TNC for short) of π, and proved several results and the applications.

- The aim of the present talk:

we will estimate the essential spectral radius $r_{ess}(\pi)$ of π for $p = 1, \infty$ by using the tail norms of π^n with $1 \le n < \infty$, i.e. the tail radius $r_{tail}(\pi)$ of π defined in the below.

• Some related facts:

(1) In order to study many interesting infinite dimensional positive operators from Euclidean quantum fields and statistical mechanics, E.Nelson [7](1966) and L. Gross in [5] (1972) introduced the conceptions of hypercontractivity and hyperboundedness of positive operators respectively.

(2) In order to study the necessary and sufficient conditions for large deviations of Markov processes on infinite dimensional spaces, Wu in [C.R. Acad. Sci. Pairs, t.321, Série I(1995), 777-762, [10]] and [J. Funct. Anal. 172(2000), 301-376 , [11]] introduced the conceptions of uniform integrability of positive operators, which are weaker than the hypercontractivity and the hyperboundedness.

For the study of essential spectrum by using the uniform integrability and the tail norm,

(3) on the one hand, by using functional inequalities, Gong and Wang in [Forum Math. 14(2002), 293–313, [1]] studied the essential spectrum of uniformly integrable semigroups and gave an estimate of essential spectrum when the semigroups have densities with respect to μ, which are the extensions of corresponding results in [J. Funct. Anal. 170(2000), 219–245, [8]]. These results were then extended to more general cases by Gong and Wang in [2] and by Wang in [9].

(4) On the other hand,

(i). for $1 < p < \infty$ we have pointed out in Remark 2.5 of [4] and *Remarks(5.ii)* of [12] that: the essential spectral radius $r_{ess}(\pi)$ of π with a density with respect to μ is equal to the tail radius $r_{tail}(\pi)$ of π , and in general case $r_{ess}(\pi) \neq r_{tial}(\pi)$.

(ii). Furthermore, Wu in [Probab. Theory Relat. Fields 128(2004), 255-321, [12]] estimated the essential spectral radius $r_{ess}(\pi)$ of π by using the Persson type's principle for $1 < p \leq \infty$ (see Theorem 5.1 in [12]), gave several results for the essential spectral radius $r_{ess}(\pi|_{b\mathcal{B}})$ of a bounded nonnegative kernel π on $b\mathcal{B}$ by using two new measures of non-compactness $\beta_\tau(\pi)$ and $\beta_w(\pi)$ which are the analogies of tail norms (see Section 3 of [12]), and proved in Theorem 3.5 of [12] that

$$r_{ess}(\pi|_{b\mathcal{B}}) = \inf_{n\geq 1} \beta_w(\pi^n)^{\frac{1}{n}}$$

under the so called condition (A1). Where $b\mathcal{B}$ consists of all bounded $\mathcal{B}$-measurable functions on E endowed the sup-norm.

(iii). Recently, H. Hennion in [6] proved that

$$r_{ess}(\pi|_{b\mathcal{B}}) = \inf_{n\geq 1} \Delta(\pi^n)^{\frac{1}{n}}$$

(see Theorem III.3 and Corollary III.6 in [6]), which is an improvement of Corollary 3.6 in [12] (see Remark III.2 in [6]). Note that, $\Delta(\pi)$ (see Definition III.4 and Corollary III.6 in [6]) is also an analogy of the tail norm.

2. The main result

Definition 0.1 *For* $p \in [1, +\infty]$ *and a bounded linear operator* $\pi : L^p \to L^p$, *we define the tail norm* $\|\pi\|_{tail(L^p)}$ *of* π *by*

(i). $\|\pi\|_{tail(L^p)} := \lim_{L\to\infty} \sup_{f:\|f\|_p \le 1} \|1_{[|\pi f|>L]}\pi f\|_p$, $\forall 1 \le p < \infty$;

(ii).

$$\begin{aligned} &\|\pi\|_{tail(L^\infty)} \\ &:= \limsup_{\mu(A)\to 0} \|\pi 1_A\|_\infty \\ &:= \lim_{\varepsilon\to 0+} \sup_{A\in\mathcal{B}:\ \mu(A)\le\varepsilon} \|\pi 1_A\|_\infty\ ; \end{aligned} \tag{0.1}$$

and we define the tail radius $r_{tail(L^p)}(\pi)$ *of* π *by*

(iii). $r_{tail(L^p)}(\pi) := \lim_{n\to\infty} \|\pi^n\|_{tail(L^p)}^{\frac{1}{n}}$.

The main result is as follows:

Theorem 0.1 *For an nonnegative operator $\pi : L^p \to L^p$ with $p = 1, \infty$ (if $p = \infty$, π is also a kernel operator, i.e., has a kernel realization) we have*

$$r_{ess}(\pi|_{L^1}) = r_{tail(L^1)}(\pi),$$
$$r_{ess}(\pi|_{L^\infty}) = r_{tail(L^\infty)}(\pi). \tag{0.2}$$

Remark 0.1 *The above results can be considered as the Gelfand Nussbaum type formulaes on L^1 and L^∞. If we introduce the tail norm condition (TNC for short) for a positive operator π in the $p = 1, \infty$ case following Definition 2.1 in [4], i.e. $r_{tail(L^p)}(\pi) < r_{sp}(\pi)$ for $p = 1, \infty$ (if $p = \infty$, π is also a kernel operator), then TNC is equivalent to $r_{sp}(\pi)$ is an isolated point in $\sigma(\pi)$ (i.e., the existence of spectral gap).*

In order to prove the main result, we need to prove the following key lemmas:

Lemma 0.2 *For an nonnegative operator $\pi : L^p \to L^p$ with $p = 1, \infty$ (if $p = \infty$, π is also a kernel operator)*

(a) the tail norm of π in L^1 has the following expressions:

$$\begin{aligned} \|\pi\|_{tail(L^1)} &= \limsup_{\mu(A)\to 0} \|1_A \pi\|_{1,1} := \lim_{\varepsilon\to 0+} \sup_{\mu(A)\leq\epsilon} \|1_A\pi\|_{1,1} \\ &= \limsup_{\mu(A),\ \mu(B)\to 0} \|1_A \pi(1_B \cdot)\|_{1,1} \\ &:= \lim_{\varepsilon\to 0+} \sup_{\mu(A),\ \mu(B)\leq\epsilon} \|1_A \pi(1_B\cdot)\|_{1,1}; \end{aligned} \tag{0.3}$$

(b) $\|\pi\|_{tail(L^1)} = \|\pi^*\|_{tail(L^\infty)}$;

(c) for two nonnegative operators π_1, π_2 on $L^p(\mu)$ (if $p = \infty$, they are also kernel operators) and $a, b \geq 0$,

$$\|\pi_1\pi_2\|_{tail(L^p)} \leq \|\pi_1\|_{tail(L^p)} \cdot \|\pi_2\|_{tail(L^p)},$$

$$\|a\pi_1 + b\pi_2\|_{tail(L^p)} \leq a\|\pi_1\|_{tail(L^p)} + b\|\pi_2\|_{tail(L^p)}.$$

Lemma 0.3 *Let* π *be a positive kernel operator on* L^∞ *and* $r_{tail(L^\infty)} < \rho$*. Then there is an integer* $l_\rho \geq 1$ *such that, for any* $l \geq l_\rho$ *there exist a positive operator* K *with the bounded density with respect to* μ *and a positive kernel operator* S *with* $\|S\|_{\infty,\infty} < \rho^l$ *satisfying*

$$\pi^l = T + S.$$

Remark 0.2 *In the proof of Lemma 0.3 we have proved that: there is a* $E_0 \in \mathcal{B}$ *with* $\mu(E_0) = 1$ *such that, for all* $l \geq$ *some integer,*

$$\sup_{\mu(B)\leq\frac{\eta}{2},\ B\in\mathcal{B}\cap E_0}\ \sup_{x\in E_0} \pi^l(x,B) < \rho^l. \tag{0.4}$$

the above (0.4) is just a kind of the Doelin's condition with respect to μ *on* E_0 *(see (1.4) in [12] and* $(\mathcal{D})$ *in [6]) for the bounded positive kernel* $\pi(x, dy)$ *on* $(E_0, \mathcal{B} \cap E_0)$. *Hence, we has essentially proved that, for a positive kernel operator* π *on* L^∞, $r_{tail(L^\infty)}(\pi) < \rho$ *implies that, there exists a subset* E_0 *with* $\mu(E_0) = 1$ *such that,* π *satisfies the Doelin's condition with respect to* μ *on* E_0 *along with the up-bound parameter* ρ. *This result was inspired by H. Hennion's Lemma 3.4 in [6].*

3. The applications

- The essential spectral radius of a symmetric positive operator

Let π be a symmetric positive operator on L^2. Then there is a μ-symmetric nonnegative kernel realization $\pi(x, dy)$ on $(E, \mathcal{B})$ with $\mu\pi \ll \mu$, which is unique up to $\mu - a.e.\ x \in E$. Suppose that the kernel $\pi(x, dy)$ is bounded, then it determines a positive operator on L^p with $1 \leq p \leq \infty$, we denote it by $\pi|_{L^p}$. For $r_{ess}(\pi|_{L^p})$ we have the following result:

Proposition 0.4 *For any* $1 < p < \infty$

$$r_{ess}(\pi|_{L^p}) \leq r_{tail(L^1)}(\pi|_{L^1}) = r_{tail(L^\infty)}(\pi|_{L^\infty}). \tag{0.5}$$

- The essential spectral radius of a positive Feller kernel operator

Let $C_b := C_b(E)$ be the Banach space of all bounded continuous functions on E endowed the sup-norm, and π be a red positive Feller kernel operator on L^∞, i.e. π is a positive kernel operator and $\pi(C_b(E)) \subseteq C_b(E)$. Note that, in this case $\pi|_{C_b(E)}$ is also a bounded linear operator on $C_b(E)$, denotes its norm by $\|\cdot\|_{C_b(E)}$, and $\mu\pi \ll \mu$. For the essential spectral radius of π we can prove that

Proposition 0.5 *Suppose that $Supp(\mu) = E$. Then for a positive Feller kernel operator π on L^∞ we have*

$$r_{tail(L^\infty)}(\pi) = r_{ess}(\pi|_{C_b(E)}). \tag{0.6}$$

Corollary 0.6 *Suppose that $Supp(\mu) = E$, and the positive Feller operator π satisfying that*

- $r_{tail(L^\infty)}(\pi) < r_{sp}(\pi)$ *(i.e., TNC),*
- π *is topologically transitive,*

where the topologically transitivity of π means that for any $x \in E$ and any nonempty open subset $O \subset E$ there is an integer $N \geq 1$ satisfying that $\pi^N(x, O) > 0$ for the Feller kernel $\pi(x, dy)$ of π. Then: π is erdogic in $C_b(E)$ and L^∞, and

$Ker(r_{sp}(\pi) - \pi)$ (resp. $Ker(r_{sp}(\pi) - \pi^*)$) is spanned by an unique $\phi > 0$ on E with $\|\phi\|_{C_b(E)} = 1$ (resp. an unique probability measure ν on $(E, \mathcal{B})$), there exist an integer d (called the period of π), a partition $\{E_j \in \mathcal{B}_\mu^+ : \ E_j \ is \ closed \ in \ E, \ j = 0, \cdots, d-1\}$ of E (called the cyclic classes of π), and $\delta, C > 0$, such that for all $k, n \in \mathbb{N}$ and for all $f \in C_b(E)$

$$\left\| \left((r_{sp}(\pi))^{-1} \pi \right)^{nd+k} f - \sum_{j=1}^{d} 1_{E_{j-k (mod \ d)}} \phi \frac{\langle 1_{E_j} f \rangle_\nu}{\langle 1_{E_j} \phi \rangle_\nu} \right\|_{C_b(E)} \tag{0.7}$$
$$\leq C e^{-\delta n} \|f\|_{C_b(E)} \, .$$

• The inequalities of essential spectral radiuses of positive bounded kernels on $b\mathcal{B}$

Let π be an nonnegative bounded kernel on $(E, \mathcal{B})$, and $b\mathcal{B} := b\mathcal{B}(E)$ denote the Banach space of all bounded $\mathcal{B}$-measurable functions on E endowed the sup-norm. Then π will determine an uniquely bounded positive operator on $b\mathcal{B}$, and we denote it by $\pi|_{b\mathcal{B}}$.

Note that, for any probability measure μ on $(E, \mathcal{B})$ with $\mu\pi \ll \pi$ the kernel π will also determine an uniquely positive operator on $L^p(\mu)$ with $1 \leq p \leq \infty$, we denote it by $\pi|_{L^p(\mu)}$. So, we can define $\sup_{\mu:\ \mu\pi\ll\mu} r_{tail(L^\infty(\mu))}(\pi)$. By the main result it is just $\sup_{\mu:\ \mu\pi\ll\mu} r_{ess}(\pi|_{L^\infty(\mu)})$.

What is the relation between it and the essential spectral radius of $\pi|_{b\mathcal{B}}$?

In the following we will try to answer this question.

Let $\mathcal{M}_b := \mathcal{M}_b(E)$ (resp. $\mathcal{M}_+ := \mathcal{M}(E)$, $\mathcal{P} := \mathcal{P}(E)$) be the space of all σ-additive signed (resp. nonnegative, probability) measures with bounded variations on $(E, \mathcal{B})$. Note that $\mathcal{M}_b$ endowed the variation norm $\|\cdot\|_{var}$ is a Banach space and a closed subspace of the dual space $(b\mathcal{B})^*$ of $b\mathcal{B}$.

Definition 0.2 *(see Definition 3.1 in [12] and Definition III.4 in [6]) For any nonnegative bounded kernel π on $(E, \mathcal{B})$ we define the following semi-norms of π:*

- $\beta_\tau(\pi|_{b\mathcal{B}}) = \sup_{(A_n)_{n\geq 1}\subset\mathcal{B}:\ A_n\downarrow\emptyset} \lim_{n\to\infty} \|\pi 1_{A_n}\|_{sup}$;
- $\Delta(\pi|_{b\mathcal{B}}) = \inf_{\nu\in\mathcal{P}} \limsup_{A\subset\mathcal{B}:\ \nu(A)\to 0} \|\pi 1_A\|_{sup}$;

and we define

- $r_\tau(\pi|_{b\mathcal{B}}) = \lim_{n\to\infty} \beta_\tau(\pi|^n_{b\mathcal{B}})^{\frac{1}{n}} = \inf_{n\geq 1} \beta_\tau(\pi|^n_{b\mathcal{B}})^{\frac{1}{n}}$;
- $r_\Delta(\pi|_{b\mathcal{B}}) = \lim_{n\to\infty} \Delta(\pi|^n_{b\mathcal{B}})^{\frac{1}{n}} = \inf_{n\geq 1} \Delta(\pi|^n_{b\mathcal{B}})^{\frac{1}{n}}$.

Proposition 0.7 *For any nonnegative bounded kernel* π *on* $(E, \mathcal{B})$ *we have*

$$\sup_{\mu:\ \mu\pi\ll\mu} r_{ess}(\pi|_{L^\infty}) = \sup_{\mu:\ \mu\pi\ll\mu} r_{tail(L^\infty(\mu))}(\pi)$$
$$\le r_\tau(\pi|_{b\mathcal{B}}) \le r_\Delta(\pi|_{b\mathcal{B}}) = r_{ess}(\pi|_{b\mathcal{B}}). \tag{0.8}$$

Corollary 0.8 *For any nonnegative bounded Feller kernel* π

$$\begin{aligned} r_{ess}(\pi|_{C_b(E)}) &= \sup_{\mu:\ \mu\pi\ll\mu,\ Supp(\mu)=E} r_{ess}(\pi|_{L^\infty(\mu)}) \\ &= \sup_{\mu:\ \mu\pi\ll\mu,\ Supp(\mu)=E} r_{tail(L^\infty(\mu))}(\pi) \\ &= r_\tau(\pi|_{b\mathcal{B}}) = r_\Delta(\pi|_{b\mathcal{B}}) = r_{ess}(\pi|_{b\mathcal{B}}). \end{aligned} \tag{0.9}$$

It follows from the above Corollary that the Conjecture in *Remarks (3.iii)* in Wu [12] holds for any nonnegative bounded Feller kernel. Note that, the estimate of $r_\tau(\pi|_{b\mathcal{B}})$ are more easier than that of $r_\Delta(\pi|_{b\mathcal{B}})$, and there are several useful criterions to estimate $r_\tau(\pi|_{b\mathcal{B}})$ for example the so called the method of Lyapunov functions see Wu [12].

Some references

1. F.Z. Gong and F.Y. Wang, Functional inequalities for uniformly integrable semigroups and application to essential spectrum, Forum Math. 14(2002), 293–313.

2. F.Z. Gong and F.Y. Wang, On Gromov's theorem and L^2 -Hodge decomposition, Int. J. Math. Math. Sci. 2004, no. 1-4, 25–44.

3. F.Z. Gong and L.M. Wu, Spectral gap of positive operators and applications, C.R. Acad. Sci. Pairs, t.331, Série 1(2000), 983-988

4. F.Z. Gong and L.M. Wu, Spectral gap of positive operators and applications, J. Math. Pures Appl. 85(2006),151-191

5. L. Gross, Existence and uniqueness of physical ground states, J. Funct. Anal. 10(1972), 52-109

6. H. Hennion, Quasi-compactness and absolutely continuous kernels, Preprint (2006).

7. E. Nelson, A quartic interaction in twodimensions, *” Mathematical theory of elementary particles”*, eds. R. Goodman and I. Segal, MIT Press, Combridge, 1966.

8. F. Y. Wang, Functional inequalities for empty essential spectrum, J. Funct. Anal. 170(2000), 219–245.

9. F.Y. Wang, Functional inequalities on Hilbert spaces and applications, Math. Z. 246(2004), 359–371.

10. L.M. Wu, Large deviations for Markov processes under superboundedness, C.R. Acad. Sci. Pairs, t.321, Série I(1995), 777-762

11. L.M. Wu, Uniformly integrable operators and large deviations for Markov processes, J. Funct. Anal. 172(2000), 301-376

12. L.M. WU, Essential spectral radius for Markov semigroups (I): discrete time case, Probab. Theory Relat. Fields 128(2004), 255-321

Thanks!