The heat-flow to equilibrium

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Götze & Bhattacharya

Time scales for Gaussian approximations and its breakdown under a hierarchy of periodic spatial heterogeneities, Bernoulli (1995)

$$dX_t^{(n)} = b(X_t^{(n)})\,dt + \beta(X_{t/n}^{(n)})\,dt + dB_t$$

b, β periodic, smooth vectorfields, $\underline{\operatorname{div}(b) = \operatorname{div}(\beta) = 0}$

$$T_t^{(n)} f(x) := E^x[f(X_t^{(n)})] \quad , t \geq 0, \; f \in \mathcal{C}_b(\mathbb{R}^d) \text{ periodic}$$

Ergodic Theorem : $\exists\, K_n, \lambda_n > 0 \; : \; \forall f \in \mathcal{C}_b(\mathbb{R}^d)$ periodic

$$\| T_t^{(n)} f - \int_{[0,1]^d} f\,dx \|_{\sup} \leq K_n e^{-\lambda_n t} \| f - \int_{[0,1]^d} f\,dx \|_{\sup}$$

Question : • Does there exist uniform constants $K, \lambda > 0$?

• What are the geometric data needed for $K, \lambda > 0$?

Generalization:

M compact Riemannian manifold, Δ Laplace-Beltrami operator

b smooth vectorfield s.th. $\mathrm{div}(b) = 0$.

$$A_b f := \Delta f + b \cdot \nabla f \quad , \; f \in \mathcal{C}^\infty(M)$$

$$T_t^{(b)} f(x) = \int_M p_b(t,x,y)\, f(y)\, dy \quad , \; t \geq 0, \; f \in \mathcal{C}(M)$$

Theorem (F04): $\exists\, K, \lambda > 0 \; : \; \forall f \in \mathcal{C}(M)$

$$\| T_t^{(b)} f - \int_M f\, dx \|_{\sup} \leq K e^{-\lambda t} \| f - \int_M f\, dx \|_{\sup}$$

$$\forall\, b \, ; \; \mathrm{div}(b) = 0 \; , \; t \geq 0$$

Special case of Bhattacharya & Götze:

$M = T^d$ d-dimensional Torus

$$A_m f = \Delta f + b \cdot \nabla f + \beta_m \cdot \nabla f \quad , \; \beta_m(x) := \beta(\tfrac{x}{m})$$

$$\mathrm{div}(b + \beta_m) = 0$$

$$\mathcal{J}_t(c) := \sup_{\mu(A)=c} \int_A T_t^{(b)} f \, dx \quad , \; f \in \mathcal{C}^\infty(M) \, , \int_M f \, dx = 0$$

- $\mathcal{J}_t(c)$ quantifies how close $T_t f$ is to the equilibrium!

- $\exists \; A_{t,c} \; : \; \mathcal{J}_t(c) = \int_{A_{t,c}} T_t^{(b)} f \, dx$

- $$\partial_t \Big|_{t=0} \mathcal{J}_t(c) = \partial_t \Big|_{t=0} \int_{A_{t,c}} f \, dx + \int_{A_{t,c}} \partial_t \Big|_{t=0} T_t^{(b)} f \, dx$$

$$= 0 + \int_{A_{t,c}} \Delta f \, dx + \int_{A_{t,c}} b \cdot \nabla f \, dx$$

$$= - \int_{\partial A_{t,c}} \langle \nu, \nabla f \rangle \, d\omega - \int_{\partial A_{t,c}} f \langle \nu, b \rangle \, d\omega$$

$$= - \int_{\partial A_{t,c}} |\nabla f| \, d\omega - 0$$

Conclusion: to have slow convergence toward equilibrium, we need small level-sets and small gradients!

- the isoperimetric function

$$\mathcal{L}(c) := \inf_{\mu(A)=c} \omega(\partial A_c)$$, μ Riemannian volume

ω d-1 dim Hausdorf measure

- the comparison manifold M^* (Berard 1986)

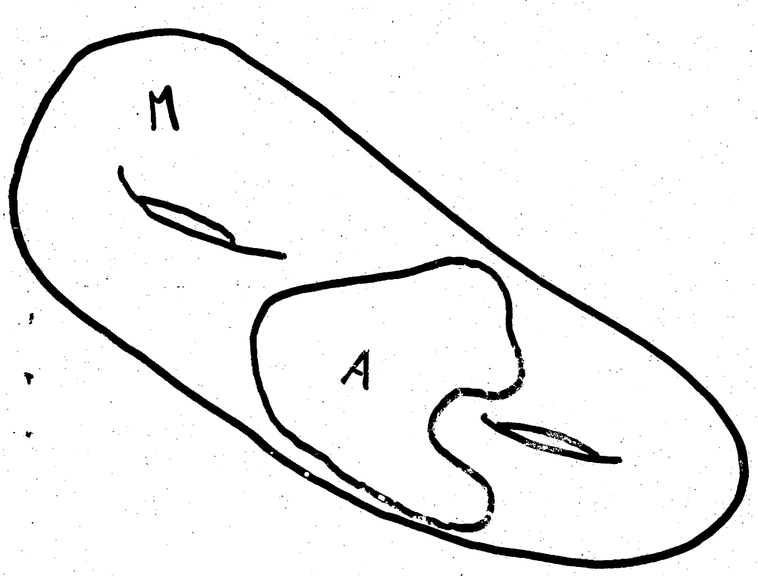
$M^* = S^d \subset \mathbb{R}^{d+1}$ with metric g^* s.th. :

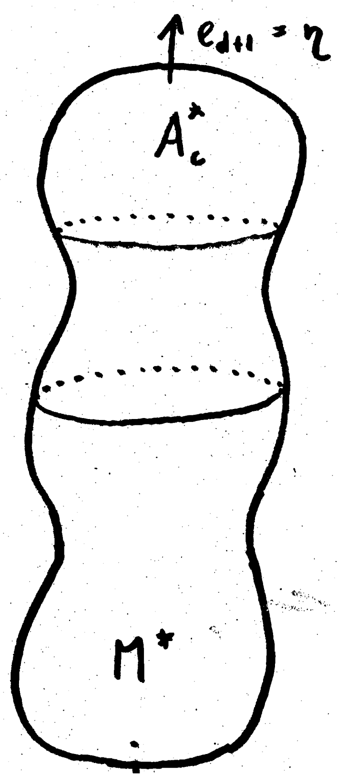
1. $\mu^*(M^*) = \mu(M)$

2. g^* is invariant with resp. to rotations fixing e_{d+1}

3. $\forall c>0$: $\inf_{\mu(A)=c} \omega(\partial A) \geq \omega^*(\partial A_c^*)$,

where A_c^* is a ball around the north-pole $e_{d+1} \in S^d$ with $\mu^*(A_c^*) = c$.





- $f \in L^1(M) \Rightarrow \exists\, f^* \in L^1(M^*)$ rearrangement of f

1. $\mu \circ f^{-1} = \mu^* \circ f^{*-1}$ i.e. f, f^* have identical distribution
2. f^* is invariant with resp. to rotations fixing e_{n+1}
3. the values of f^* decrease with the distance to η.

Proposition: $f \in \mathcal{C}^1(M) \Rightarrow$

$$\int_{\partial A_c} |\nabla f| \, d\omega \geq \int_{\partial A_c^*} |\nabla f^*| \, d\omega \quad \text{for almost all } c.$$

(whenever $f(\partial A_c)$ is a regular value)

Consequence: the heat dissipation on M can be controled by suitable heat-distributions on M^*.

Theorem: $\forall c > 0 : \forall f \in L^1(M)$

$$\sup_{\mu(A)=c} \int_A T_t f \, dx \leq \int_{A_c^*} T_t^* f^* \, dx$$

T_t heat-semigroup on M

T_t^* heat-semigroup on M^*.

- ϕ_t flow of the ODE $\dot{x} = b(x)$

$S_t^{(b)} f(x) := f \circ \phi_t(x)$ is group of transformations on $\mathcal{C}(M)$

$Bf := b \cdot \nabla f$ is generator of S_t on $\mathcal{C}^\infty(M)$

- Trotter product formula : $\forall f \in \mathcal{C}(M)$:

$$T_t^{(b)} f = \lim_{n\to\infty} \left(T_{\frac{t}{n}} S_{\frac{t}{n}}^{(b)}\right)^n f \quad \text{with resp. to sup-norm}$$

$\Rightarrow$

$$\int_A T_t^{(b)} f \, dx \leq \int_A \left(T_{\frac{t}{n}} S_{\frac{t}{n}}^{(b)}\right)^n f \, dx + \varepsilon$$

$$\leq \int_{A_c^*} T_{\frac{t}{n}}^* \left(S_{\frac{t}{n}}^{(b)} \left(T_{\frac{t}{n}} S_{\frac{t}{n}}\right)^{n-1} f \right)^* dx + \varepsilon$$

$$= \int_{A_c^*} T_{\frac{t}{n}}^* \left(\left(T_{\frac{t}{n}} S_{\frac{t}{n}}\right)^{n-1} f \right)^* dx + \varepsilon$$

$$\overset{\text{induction}}{\leq} \int_{A_c^*} T_{\frac{t}{n}}^* \left(T_{\frac{t}{n}}^*\right)^{n-1} f^* \, dx + \varepsilon$$

$$= \int_{A_c^*} T_t^* f^* \, dx + \varepsilon$$

$$\Rightarrow \quad \sup_{\mu(A)=c} \int_A T_t^{(b)} f \, dx \quad \leq \quad \int_{A^*} T_t^* f^* \, dx$$

$$\mu(A_\varepsilon) \downarrow 0 \quad , \quad x_0 \in A_\varepsilon$$

$$\Rightarrow \quad T_t^{(b)} f(x_0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\mu(A_\varepsilon)} \int_{A_\varepsilon} T_t^{(b)} f(x) \, dx$$

$$\leq \lim_{\varepsilon \downarrow 0} \frac{1}{\mu(A_\varepsilon^*)} \int_{A_\varepsilon^*} T_t^* f(x) \, dx \quad = T_t^* f(n)$$

$$\Rightarrow \quad \sup_{x \in M} T_t^{(b)} f(x) \quad \leq \quad T_t^* f(n)$$

With similar arguments : $\displaystyle \inf_{\mu(A)=c} \int_A T_t^{(b)} f \, dx \geq \int_{A_*} T_t^* f^* \, dx$

where A_* is a ball around south-pole s with $\mu^*(A_*) = c$

$$\Rightarrow \quad \inf_{x \in M} T_t^{(b)} f(x) \quad \geq \quad T_t^* f^*(s)$$

$$\Rightarrow \quad \| T_t^{(b)} f \|_{sup} \quad \leq \quad \| T_t^* f^* \|_{sup}$$

$$\leq K^* e^{-\lambda^* t} \| f^* \|_{sup} \quad = K^* e^{-\lambda^* t} \| f \|_{sup}$$