Monge Optimal Transport and Fokker-Planck Equations

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July, 18, 2007

Let X = Polish space with distance d. Given Probability measures μ and ν, the L^2-Wasserstein distance is defined as

$$W_2^2(\mu,\nu) = \inf_{\pi \in \mathscr{C}(\mu,\nu)} \int_{X\times X} d(x,y)^2\, \pi(\mathrm{d}x, \mathrm{d}y).$$

In general, W_2 is a pseudo-distance. Let

$$\mathscr{P}_2(X) = \Big\{ \mu \Big| M_2(\mu) := \int_X d(x_0,x)^2\, \mathrm{d}\mu(x) < +\infty \Big\},$$

where x_0 is fixed. It's known that $(\mathscr{P}_2(X), W_2)$ is a Polish space.

Moreover, $\mu_n \to \mu$ w.r.t. W_2

$\iff$ $\mu_n \to \mu$ weakly and

$\lim_{N\to\infty} \sup_n \int_{d(x_0,x)\geq N} d(x_0,x)^2 \mathrm{d}\mu_n(x) = 0.$

Monge optimal transport is to find $T : X \to X$ which pushes μ forward to ν, such that

$$W_2^2(\mu,\nu) = \int_X d(x, T(x))^2 \, \mathrm{d}\mu(x).$$

Here are some examples for which T does exist.

- $X = \mathbb{R}^d$, $\mu \ll \lambda_d,\ \nu \ll \lambda_d$, $\exists\ !T$ (Brenier, 91).
- $X = \mathbb{R}^d$, $\mu,\ \nu$ not charge subset of $\dim_H < d$ (McCann, 96).
- $X =$compact manifold, (McCann 2001) J. Geometric Analysis.

- X =Wiener space=$C_0([0,1],\mathbb{R})$, μ=Wiener measure. The Riemannian distance is

$$d_H(x,y) = \begin{cases} |x-y|_H, & \text{if } x-y \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here $h \in H \iff \int_0^1 |\dot{h}(t)|^2 \, \mathrm{d}t < +\infty$. The moment of order 2

$$\int_X |x - x_0|^2 \, \mathrm{d}\nu(x) < +\infty$$

is dependent of $x_0 \in X$. so the space $\mathscr{P}_2(X)$ is not suitable.

The following result is due to Feyel and Üstünel (PTRF 2005).

Theorem *There exists a unique $S : X \to H$ such that*

$$(I + S)_* f\mu = g\mu$$

and $\int_X |S(x)|_H^2 \, f(x)\mathrm{d}\mu(x) = W_2^2(f\mu, g\mu)$ *if* $\mathrm{Ent}_\mu(f) < +\infty$, $\mathrm{Ent}_\mu(g) < +\infty$, *where* $\mathrm{Ent}_\mu(f) := \int f \log f \, \mathrm{d}\mu$.

- Another example is $X =$loop group

$$\mathscr{L}_e(G) = \{\gamma : [0,1] \to G \text{ continuous}; \ \gamma(0) = \gamma(1) = e\}.$$

It is the simplest non flat infinite dimensional manifold. (Fang and J. Shao, JFA 2007).

Fokker-Planck Equations Let's regard the case

$$\frac{\partial \rho_t}{\partial t} = -L\rho_t + \mathrm{div}_\gamma(\nabla\psi \cdot \rho_t) \qquad (FP)$$

where (ρ_t) is a family of probability measures on $\mathbb{R}^d$, γ is the standard Gaussian measure on $\mathbb{R}^d$, and $\mathrm{div}_\gamma(Z)$ is defined by

$$\int_{\mathbb{R}^d} \varphi \mathrm{div}_\gamma(Z) \mathrm{d}x = -\int_{\mathbb{R}^d} \langle \nabla\varphi, Z \rangle \, \mathrm{d}x.$$

The equation is understand in distribution sense:

$$-\int_{[0,\infty[\times\mathbb{R}^d} \alpha'(t)\rho_t\varphi(x)\,\mathrm{d}t\mathrm{d}\gamma(x)$$
$$= \int_{\mathbb{R}^d} \rho_0\alpha(0)\varphi(x)\,\mathrm{d}\gamma(x) + \int_{[0,\infty[\times\mathbb{R}^d} \alpha(t)\rho_t\big(L\varphi - \langle\nabla\varphi, \nabla\psi\rangle\big)\,\mathrm{d}t\mathrm{d}\gamma$$

for $\alpha \in C_c^\infty([0,\infty[)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$, where $\phi \geq 0$ is given but we do not assume that

$$\int e^{\lambda\phi}\,\mathrm{d}\gamma < +\infty.$$

We can not apply the theory of Dirichlet form.

De Giorgin's minimizing motive Let (E, d) be a metric space. $\phi : E \to]-\infty, +\infty]$ be a "good function". The method consists of two steps

(i) given $h > 0$, define $U_h^{(n)} \in E$ which minimizes

$$U \mapsto \frac{1}{2} d^2(U, U_h^{(n-1)}) + h\Phi(U).$$

(ii) define $U_h(t) = \sum_{n=0}^{\infty} U_h^{(n)} \mathbf{1}_{[nh,(n+1)h[}(t)$.

Then as $h \downarrow 0$, $U_h(t)$ converges to the flow associated to the gradient $-\nabla\Phi$:

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = -\nabla\Phi(U_t) \qquad \text{(in good case)}.$$

Wasserstein distance approach by Jordan, Kinderlehrer, Otto(98)

Consider $\mathscr{K} = \{\rho : \mathbb{R}^d \to \mathbb{R}_+ \big| \int \rho \,\mathrm{d}\gamma = 1,\ M_2(\rho) < +\infty\}$.

For $\rho_0 \in \mathscr{K}$ such that $\Phi(\rho_0) < +\infty$, where

$$\Phi(\rho) = \int \phi\rho \,\mathrm{d}\gamma + \mathrm{Ent}_\gamma(\rho).$$

(i) $\exists\, !\rho^{(n)} \in \mathscr{K}$ such that

$$\frac{1}{2}W_2^2(\rho^{(n-1)}, \rho^{(n)}) + h\Phi(\rho^{(n)}) = \inf_{\rho\in\mathscr{K}} \left\{\frac{1}{2}W_2^2(\rho^{(n-1)}, \rho) + h\Phi(\rho)\right\}.$$

(ii) $\rho_h(t) = \sum_{n=0}^{\infty} \rho^{(n)} \mathbf{1}_{[nh,(n+1)h[}(t) \to \rho(t)$ which is a solution of (FP).

Why and what is "$\nabla\Phi$"?

Let Z be a smooth vector field on $\mathbb{R}^d$, with compact support. (U_t^Z) flow of diffeomorphism. Then

$$(U_t^Z)_*\gamma = k_t\gamma,\ k_t(x) = \exp\Big(-\int_0^t \mathrm{div}_\gamma(Z)(U_{-s}^Z)\,\mathrm{d}s\Big).$$

Recall that $\int \mathrm{div}_\gamma(Z)\cdot\phi\,\mathrm{d}\gamma = -\int\langle\nabla\phi, Z\rangle\,\mathrm{d}\gamma$. Let $\rho\in\mathscr{K}$. Consider

$$\rho_t = (U_t^Z)_*(\rho\gamma).$$

We have

$$\int f\ \mathrm{d}\rho_t = \int f(U_t^Z)\rho\,\mathrm{d}\gamma = \int f\rho(U_{-t}^Z)k_t\,\mathrm{d}\gamma.$$

So $\rho_t = \rho(U_t^Z)k_t$.

$$\begin{aligned}\mathrm{Ent}_\gamma(\rho_t) &= \int \rho(U_{-t}^Z)k_t \log\left(\rho(U_{-t}^Z)\right) \mathrm{d}\gamma \\ &= \int \rho \log\left(\rho k_t(U_t^Z)\right) \mathrm{d}\gamma \\ &= \mathrm{Ent}_\gamma(\rho) + \int \log k_t(U_t^Z)\ \rho \mathrm{d}\gamma\end{aligned}$$

So $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ent}_\gamma(\rho_t) = -\int \mathrm{div}_\gamma(Z)\rho\, \mathrm{d}\gamma$. Next

$$\int \psi\rho_t\ \mathrm{d}\gamma = \int \psi\rho(U_{-t}^Z)k_t\, \mathrm{d}\gamma = \int \psi(U_t^Z)\rho\, \mathrm{d}\gamma.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \psi\rho_t\, \mathrm{d}\gamma = \int \langle \nabla\psi, Z\rangle \rho\ \mathrm{d}\gamma.$$

Now let $Z = \nabla\varphi,\ \varphi \in C_c^\infty(\mathbb{R}^d)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(\rho_t) = -\int L\varphi\rho\,\mathrm{d}\gamma + \int \langle\nabla\psi, \nabla\varphi\rangle\rho\,\mathrm{d}\gamma.$$

Otto "Tangent space" For $\rho \in \mathscr{K}$,

$$T_\rho\mathscr{K} = \text{generated by } \{\nabla\varphi;\ \varphi \in C_c^\infty(\mathbb{R}^d)\}.$$

<u>Sketch of the proof for (i)</u> It is sufficient to consider $n = 0$. There is a sequence $(\rho_m)_{m\geq 1}$ such that

$$0 \leq \frac{1}{2}W_2^2(\rho_0, \rho_m) + h\Phi(\rho_m) \leq \inf + \frac{1}{m} < \inf + 1$$

This implies

$$(1) \quad \sup_m \int |x|^2_{\mathbb{R}^d} \, d\rho_m < +\infty \Rightarrow \{\rho_m\} \text{ is tight.}$$

In fact,

$$(2) \quad R^2 \rho_m(|x| > R) \le \int |x|^2 \, d\rho_m,$$

so $\rho_m \to \rho$ weakly as $m \to +\infty$.

Remarks towards Wiener space: (1) has no meaning; (2) $\{|x| \le R\}$ is not compact for uniform norm and too poor for $|\cdot|_H$.

Wiener space $X = C_0([0,1],\mathbb{R})$, H = Cameron-Martin space, μ = Wiener measure.

We consider

$$\mathscr{K} = \{\rho : X \to \mathbb{R}_+ \Big| \int_X \rho \, \mathrm{d}\mu = 1,\ \mathrm{Ent}_\mu(\rho) < +\infty\}.$$

Proposition 1 Let $\psi : X \to \mathbb{R}_+$. Suppose there is $\rho \in \mathscr{K}$ such that $\int_X \psi\rho \, \mathrm{d}\mu < +\infty$. Then for any $\rho_0 \in \mathscr{K}$, $h > 0$, there is a unique $\hat{\rho} \in \mathscr{K}$ such that

$$\frac{1}{2} W_2^2(\rho_0, \hat{\rho}) + h\Phi(\hat{\rho}) = \inf_{\rho \in \mathscr{K}} \Big\{ \frac{1}{2} W_2^2(\rho_0, \rho) + h\Phi(\rho) \Big\}.$$

Recall that

- $W_2^2(\rho_0, \hat{\rho}) = \inf_{\pi \in \mathscr{C}(\rho_0, \hat{\rho})} \Big\{ \int |x-y|_H^2 \, \pi(\mathrm{d}x, \mathrm{d}y) \Big\}$.
- $W_2^2(\rho_0 \mu, \mu) \le \mathrm{Ent}_\mu(\rho_0)$ by Talagrand's inequality 1996. So $W_2^2(\rho_0, \rho_1) < +\infty$ if $\rho_0, \ \rho_1 \in \mathscr{K}$.

Proof. As before, we have $\rho_m \in \mathscr{K}$ such that

$$\frac{1}{2} W_2^2(\rho_0 \mu, \rho_m \mu) + h\Phi(\rho_m) \le \inf + 1.$$

Now using Feyel-Üstünel's result, $\exists \ S_m : X \to H$ such that

$$\big(I + S_m\big)_*(\rho_0 \mu) = \rho_m \mu$$

$$W_2^2(\rho_0 \mu, \rho_m \mu) = \int_X |S_m|_H^2 \rho_0 \mathrm{d}\mu.$$

Let $\varepsilon > 0$, $\exists\ K \subset X$ compact such that $\rho_0(K^c) < \frac{\varepsilon}{2}$. Consider $K_R = K + \{h \in H \big| |h|_H \leq R\}$ compact in X. We have

$$\rho_m(K_R^c) = \int_X \mathbf{1}_{K_R^c}(x + S_m(x))\rho_0(x)\,\mathrm{d}\mu(x)$$
$$= \int_K \mathbf{1}_{K_R^c}(x + S_m(x))\rho_0\,\mathrm{d}\mu + \int_{K^c} \mathbf{1}_{K_R^c}(x + S_m(x))\rho_0\,\mathrm{d}\mu$$

The second term is $\leq \rho_0(K^c) < \frac{\varepsilon}{2}$, while the first is dominated by

$$\int_{|S_m(x)|>R} \rho_0(x)\,\mathrm{d}\mu(x) \leq \frac{1}{R^2}\int |S_m(x)|_H^2 \rho_0(x)\,\mathrm{d}\mu(x) < \frac{\varepsilon}{2}.$$

So that $\{\rho_m\mu;\ m \geq 1\}$ is tight. Up to a subsequence, $\rho_m\mu \to \nu$.

Now the condition

$$\sup_m \mathsf{Ent}_\mu(\rho_m) < +\infty$$

implies that $\{\rho_m;\ m \geq 1\}$ is uniformly integrable; so that for any bounded Borel function F

$$\lim_{m\to+\infty} \int_X F\rho_m \,\mathrm{d}\mu = \int_X F \,\mathrm{d}\nu.$$

Therefore, $\nu \ll \mu$. Let $\hat\rho = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. It is standard to prove that

$$\mathsf{Ent}_\mu(\hat\rho) \leq \liminf_{m\to\infty} \mathsf{Ent}_\mu(\rho_m).$$

In conclusion $\hat\rho \in \mathscr{K}$.

Let $\rho^{(0)} \in \mathscr{K}$ be given such that

$$\Phi(\rho^{(0)}) = \int_X \psi \rho^{(0)} \, d\mu + \mathrm{Ent}_\mu(\rho^{(0)}) < +\infty.$$

Define

$$\rho_h(t) = \sum_{k=0}^{\infty} \rho^{(k)} \mathbf{1}_{[kh,(k+1)h)}(t).$$

Proposition 2 Let $T > 0$. Then the family of measure $\{\rho_h(t) dt d\mu;\ h > 0\}$ on $[0, T] \times X$ is tight.

Proof. By construction of $\{\rho^{(k)};\ k \geq 1\}$, we have, for each $k \geq 1$,

$$\frac{1}{2} W_2^2(\rho^{(k-1)}, \rho^{(k)}) + h\Phi(\rho^{(k)}) \leq h\Phi(\rho^{(k-1)}).$$

Let $N \geq 1$ such that $Nh \leq T$.

Then

$$\frac{1}{2}\sum_{k=1}^{N} W_2^2(\rho^{(k-1)}, \rho^{(k)}) + h\Phi(\rho^{(N)}) \leq h\Phi(\rho^{(0)}).$$

Therefore for any $1 \leq q \leq N$,

$$W_2^2(\rho^{(0)}, \rho^{(q)}) \leq N \sum_{k=1}^{N} W_2^2(\rho^{(k-1)}, \rho^{(k)}) \leq 2Nh\Phi(\rho^{(0)}) \leq 2T\Phi(\rho^{(0)}).$$

In the same way, there exists a compact $K \subset X$ such that

$$\rho^{(q)}(K^c) < \varepsilon, \quad \forall 1 \leq q \leq N.$$

Therefore

$$\int_{[0,T]\times K^c} \rho_h(t)\,\mathrm{d}t\mathrm{d}\mu = \sum_{k=1}^{N} \rho^{(k)}(K^c)h \leq \varepsilon N h \leq \varepsilon T.$$

In the same way, Letting $h \downarrow 0$

$$\rho_h(t)\mathrm{d}t\mathrm{d}\mu \to \rho_t \mathrm{d}t\mathrm{d}\mu, \quad \text{weakly in } L^1.$$

Now $(\rho_h(t))$ satisfies the following (FP):

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = L\rho_t + \mathrm{div}_\mu(\nabla\psi \cdot \rho_t), \quad \rho_t\big|_{t=0} = \rho^{(0)},$$

where $L =$ O.U. operator on X.

Thank You!