Discrete Approximations to Reflected Brownian Motions

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Reflected Brownian Motion

When $D \subset \mathbf{R}^n$ is C^2:

$$X_t = X_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s,$$

B: Brownian motion in $\mathbf{R}^n$,

$\mathbf{n}$: unit inward normal at ∂D

L: boundary local time that is non-decreasing and increases only when $X_s \in \partial D$.

• The infinitesimal generator $\mathcal{L}$ is $\frac{1}{2}\Delta$ with zero Neumann condition.

Question 1: How to simulate RBM?
Question 2: How to construct RBM on non-smooth domains?

Green-Gauss formula

$$\int_D v(x)\Delta u(x)dx = -\int_D \nabla u(x)\cdot\nabla v(x)dx - \int_{\partial D} v(x)\frac{\partial u}{\partial \mathbf{n}}(x)\sigma(dx).$$

So for $u \in \mathcal{D}(\mathcal{L})$,

$$(-\mathcal{L}u, v)_{L^2(D)} = \frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)dx.$$

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Dirichlet form $(\mathcal{E}, \mathcal{F})$

$$\begin{cases} \mathcal{F} = \mathcal{D}(\sqrt{-\mathcal{L}}) = W^{1,2}(D), \\ \mathcal{E}(u,v) = (\sqrt{-\mathcal{L}}u, \sqrt{-\mathcal{L}}v)_{L^2(D)} = \frac{1}{2}\int_D \nabla u(x)\cdot\nabla v(x)dx. \end{cases}$$

Reflected BM on non-smooth domains

Fukushima (1967): X^* on Martin-Kuramochi compactification D^* of D.

Chen (1993): $X = \pi(X^*)$ projection on $\overline{D}$. X is symmetric Markov process on D but may not be strong Markov.

Chen (1993), Chen-Fitzsimmons-Williams (1993): Sample path behavior (Skorokhod decomposition)

Chen (1993): Approximations from inside by RBMs on smooth domains.

Condition

$$C^1(\overline{D}) \text{ is dense in } (W^{1,2}(D), \|\cdot\|_{1,2}) \tag{1}$$

guarantees the regularity on $\overline{D}$.

Uniform domains

Extension operator: $T : W^{1,2}(D) \to W^{1,2}(\mathbf{R}^n)$

Bounded Lipschitz domains and (ε, δ)-domains are $W^{1,2}$-extension domains.

D is an (ε, δ)-domain if for $x, y \in D$ with $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset D$ joining x and y with length$(\gamma) \le \varepsilon^{-1}|x - y|$ such that for any $z \in \gamma$,

$$\min \{|x - z|, |z - y|)\} \le \varepsilon^{-1} \text{dist}(z, \partial D).$$

Uniform domains $=(\varepsilon, \infty)$-domains.

Example: von Koch snowflake domain.

More on Uniform domains

The boundary of a uniform domain can be highly non-rectifiable and, in general, no regularity of its boundary can be inferred (besides the easy fact that the Hausdorff dimension of the boundary is strictly less than n).

For any $\alpha \in [n-1, n)$, one can construct a uniform domain $D \subset \mathbf{R}^n$ such that $\mathcal{H}^\alpha(U \cap \partial D) > 0$ for any open set U satisfying $U \cap \partial D \neq \emptyset$. Here $\mathcal{H}^\alpha$ denotes the α-dimensional Hausdorff measure in $\mathbf{R}^n$.

Benjamini-Chen-Rohde (2004): Uniform Hausdorff dimensional result for RBM on uniform domains, Hausdorff dimension on boundary occupation time and on trace on the boundary.

Discrete Approximation

Assume $0 \in D$ and $|\partial D| = 0$ in the first two approximations.

Let D_k be the connected component of $D \cap 2^{-k}\mathbf{Z}^n$ that contains 0 with edge structure inherited from $2^{-k}\mathbf{Z}^n$.

$v_k(x)$: the degree of a vertex x in D_k.

Let X^k and Y^k be the discrete and continuous time simple random walks on D_k with stationary initial distribution m_k, moving at the rate 2^{-2k}, respectively, where $m_k(x) = \frac{v_k(x)}{2n} 2^{-kn}$.

Theorem (Burdzy-C.) Under condition (1), both $\{X^k, k \geq 1\}$ and $\{Y^k, k \geq 1\}$ converge weakly to the stationary reflected Brownian motion on D in the Skorokhod space $\mathbf{D}([0,1], \mathbf{R}^n)$.

Domain regularity condition

Some domain regularity condition is needed for the above theorem to hold.

A Counter-example. Regularity conditions are needed for the random walk approximations. Let

$U_k^\varepsilon = \{(x,y) \in (0,1)^2 : |x - j2^{-k}| < \varepsilon$ or $|y - j2^{-k}| < \varepsilon$ for some $j \in \mathbf{Z}\}$.

Choose $\varepsilon_k > 0$ so that $|U_k^{\varepsilon_k}| < 2^{-k-1}$ and let $U = \bigcup_{k \geq 1} U_k^{\varepsilon_k}$. U is a bounded open connected set with Lebesgue area less than 1/2.

Note that D_k defined for U is the same as that relative to $D = (0,1)^2$.

Myopic Conditioning

The "myopic conditioning" approximation below works on any bounded domain.

For every integer $k \geq 1$, let $\{Z^k_{j2^{-k}}, j = 0, 1, 2, \dots\}$ be a discrete time Markov chain with one-step transition probabilities being the same as those for the Brownian motion in D conditioned not to exit D before time 2^{-k}. The process Z^k_t can be defined for $t \in [(j-1)2^{-k}, j2^{-k}]$ either as the conditional Brownian motion going from $Z^k_{(j-1)2^{-k}}$ to $Z^k_{j2^{-k}}$ without leaving the domain D or as a linear interpolation between $Z^k_{(j-1)2^{-k}}$ and $Z^k_{j2^{-k}}$.

Theorem (Burdzy-C.): For any bounded domain D, the laws of Z^k (defined in either way) converge to that of the reflected Brownian motion on D.

Remarks and History

The myopic conditioning approximation of reflected Brownian motion is proved for every starting point $x \in D$ so these theorems demonstrate explicitly that the symmetric reflected Brownian motion on D is completely determined by the absorbing Brownian motion in D.

History: Stroock-Varadhan (1971): discrete approximations to RBM in C^2-domains.

Varopoulos (2003): approximations to the *killed* BM in Lipschitz domains.

Our Approach

In the first two approximation schemes (i.e. random walk approximations), define $m_k(x) = \frac{v_k(x)}{2n} 2^{-kn}$ for $x \in D_k$.

In the myopic conditioning scheme, define $m_k(dx) := 1_D(x) P^D_{2^{-k}} 1(x)\, m(dx)$.

Let X^k be one of the discrete approximating processes mentioned above. Then m_k is the reversible measure for X^k in a suitable sense.

Strategy

We will show that

• the law of $\{X^k, \mathbf{P}^k_{m_k}, k \geq 1\}$ is tight in the space $C([0,1], \mathbf{R}^n)$ or $D([0,1], \mathbf{R}^n)$, and that any of its weak subsequential limits $(Z, \mathbf{P})$ is a time-homogeneous Markov process that is time-reversible with respect to the Lebesgue measure m in D.

• the process Z killed upon leaving domain D is a killed Brownian motion in D and establish that the Dirichlet form $(\mathcal{E}^Z, \mathcal{F})$ of Z in $L^2(D, m)$ has the property that

$$W^{1,2}(D) \subset \mathcal{F} \qquad \text{and} \qquad \mathcal{E}^Z(f,f) \leq \mathcal{E}(f,f) := \frac{1}{2}\int_D |\nabla f(x)|^2 dx$$

for $f \in W^{1,2}(D)$.

(continued)

We then get the desired result by the following

Theorem (Silverstein, 1974)
Let D be a bounded domain in $\mathbf{R}^n$ and m_D be the Lebesgue measure in D that is extended to $\overline{D}$ by taking $m_D(\overline{D} \setminus D) = 0$. Suppose that Z is a $\overline{D}$-valued right continuous time-homogeneous Markov process having left-limits with initial distribution m_D and is symmetric with respect to measure m_D. Let $(\mathcal{E}^Z, \mathcal{F})$ be the Dirichlet form of Z in $L^2(\overline{D}, m_D)$. If the subprocess of Z killed upon leaving domain D is a killed Brownian motion in D, then

$$\mathcal{F} \subset W^{1,2}(D) \qquad \text{and} \qquad \mathcal{E}^Z(f,f) \geq \mathcal{E}(f,f) \quad \text{for } f \in \mathcal{F}.$$

Discrete random walk approximation

- Tightness: time-reversal method, forward-martingale decomposition.
- An inequality: Let Q_k be the one-step transition of RW on D_k.

$$(f - Q_k^{2j} f, f)_{L^2(D,m_k)} \quad \leq \quad 2j(f - Q_k f, f)_{L^2(D,m_k)}.$$

- Dirichlet form comparison: Let $(\mathcal{E}, \mathcal{F})$ be the symmetric Dirichlet form of X. Then the above implies that $C^1(\overline{D}) \subset \mathcal{F}$ and

$$\mathcal{E}(f,f) \leq \frac{1}{2n} \int_D |\nabla f|^2 dx$$

for $f \in C^1(\overline{D})$. Thus under assumption (1), we have $W^{1,2}(D) \subset \mathcal{F}$.

Myopic conditioning

Let Y^k be myopic conditioning of killed BM with linear interpolation between time interval $[(j-1)2^{-k}, j2^{-k}]$.

Lemma Suppose that either (i) $\mu_k = m_k$ for every $k \geq 1$; or (ii) $\{\mu_k, k \geq 1\}$ is a sequence of measures on D with $\sup_{k\geq 1} \mu_k(D) < \infty$ and $\mu_k(D \setminus K) = 0$ for some compact subset K of D and all $k \geq 1$. Then the laws of $\{Y^k, \mathbf{P}^k_{\mu_k}, k \geq 1\}$ are tight in the space $C([0,1], \mathbf{R}^n)$.

Sketched Proof

By Ito's formula, for every $f \in C^2(\overline{D})$, $\{f(Y^k_{j2^{-k}}) + \frac{\|\Delta f\|_\infty}{2} j2^{-k}, \mathcal{G}^k_{j2^{-k}}\}_{j=0,1,\cdots,2^k}$ is a non-negative $\mathbf{P}^k_{m_k}$-submartingale. Moreover, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{j=1}^{2^k} \mathbf{P}^k_{m_k}\left(|Y^k_{j2^{-k}} - Y^k_{(j-1)2^{-k}}| > \varepsilon\right) \\
\leq\ & \lim_{k\to\infty} 2^k \int_D \mathbf{P}_x(|X_{2^{-k}} - X_0| > \varepsilon \text{ and } 2^{-k} < \tau_D)dx \\
\leq\ & \lim_{k\to\infty} 2^k \int_D \mathbf{P}_x(|X_{2^{-k}} - X_0| > \varepsilon)dx \\
=\ & 0.
\end{aligned}$$

This implies the laws of $\{Y^k, \mathbf{P}^k_{m_k}, k \geq 1\}$ are tight in $C([0,1], \mathbf{R}^n)$.

(continued)

For case (ii), we use the property of BM:

$$\mathbf{P}\left(\sup_{s\leq t}|B_s - B_0| > r\right) \leq c_0 \exp\left(-\frac{r}{c_0 t}\right).$$

and reduce the case to the stationary distribution.

Lemma. Let $(Y, \mathbf{P})$ be a weak limit of Y^k. Then for every $f \in C_c^\infty(D)$, $M_t^f := f(Y_t) - f(Y_0) - \frac{1}{2}\int_0^t \Delta f(Y_s)ds$ is a $\mathbf{P}$-square integrable martingale. This in particular implies that $\{Y_t, t < \tau_D, \mathbf{P}\}$, with $\tau_D := \inf\{t > 0 : Y_t \notin D\}$, is the killed Brownian motion in D with initial distribution μ.

Sketched proof. Prove $\mathcal{L}_k f(x) := \int_D (f(y) - f(x))Q_k(x, dy)$ converges uniformly to $\frac{1}{2}\Delta f$ for $f \in C_c^3(D)$. $\square$

Weak Convergence

A similar approach through semigroup inequality proves the weak convergence of $(Y^k, \mathbf{P}_{m_k})$ to stationary RBM on $\overline{D}$.

Weak convergence of $(Y^k, \mathbf{P}_{\mu_k})$ requires extra work. The idea is BM spread out immediately and so can reduce the case with stationary initial distribution.

The “true" myopic conditioning requires extra work in the tightness proof.

Summary

New approaches are introduced for discrete approximation of RBMs. In particular, these approaches do not require detailed heat kernel estimates and so they are potentially applicable to various other stochastic models.

These approximation schemes (especially the random walk approximations) give not only new ways of constructing RBMs but also implementable algorithms to simulate RBMs.

THE END