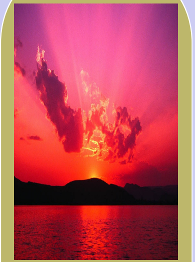

Nonlinear Expectations and Nonlinear Pricing

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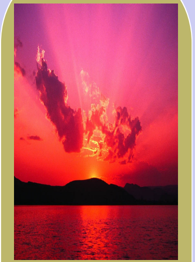


Home Page
Title Page
Page 2 of 18
Go Back
Full Screen
Close
Quit

Objective

Considering the difference among Four Modes of nonlinear expectations:

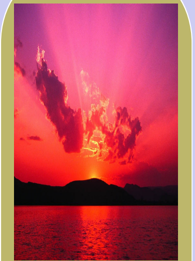
(i) Choquet expectations.

(ii) g-expectation.

(iii) Coherent risk measures.

(iv) Convex risk measures.

We will show that g-expectation is the best expectation to deal with nonlinear pricing in continuous-time setting,.



Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 3 of 18
Go Back
Full Screen
Close
Quit

Motivation

⋆ Finance

(Linear) expectation ← Black-Scholes → Complete Markets

? $\Longleftrightarrow$ Incomplete Markets.

⋆ Economics:Rational expected Utility

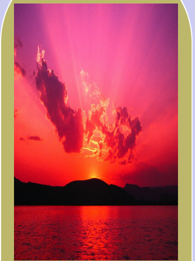
⋆ von Neumann, Morgenstern 1944:

Given "preference" $\succeq$ over acts ξ, η, where $\xi \succeq \eta$ denotes that ξ is preferred to η. Axioms P1 through P6 imply that there exists a unique finitely additive, non-atomic probability measure $P(\cdot)$ on $\mathcal{F}$, and a utility function $U(\cdot)$, such that

$$\xi \succeq \eta \Leftrightarrow E_p U(\xi) \geq E_p U(\eta).$$

Allias' Paradox (1953), Nobel prize (1988).

How about non-rational expected utility? e.g. ambiguity, uncertainty.



Motivation

$\star$ Finance

(Linear) expectation $\longleftarrow$ Black-Scholes $\longrightarrow$ Complete Markets

Nonlinear expectation $\Longleftrightarrow$ Incomplete Markets.

New research area: Behavior finance

$\star$ Economics–decision theory

(Linear) expectation $\longleftarrow$ Neumann $\longrightarrow$ Rational expected utility

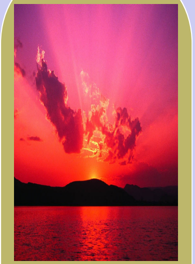
Nonlinear expectation $\Longleftrightarrow$ Non-rational expected utility.

New research area: Behavior economics.

Kahneman, Nobel prize (2002).

Remark: Probability theory is not enough to deal with incomplete markets and non-rational expected utility because of its linearity.

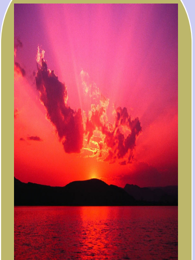
A new mathematical tool—non linear probability theory is needed in finance and economics.



Why to deal with nonlinear pricing

The current mathematical framework of financial economics is pre-dominantly linear. That is, the entire mathematical construct in financial economics is linear because the construct itself assume that the input-output relationships are proportional. Unfortunately, this is not so; the input-output relationships of financial economics are not linear. They are nonlinear, or disproportionate.

by Christopher T. May (1999) — Nonlinear Pricing: Theory and Application



Linear and nonlinear Expectation

Linear case: $(\Omega, \mathcal{F}, P)$, $E[\xi] : L^1 \to R$ linear functional.

$$E[\xi + \eta] = E[\xi] + E[\eta], \qquad \forall \xi, \eta \in L^1.$$

or $P : \mathcal{F} \to [0,1]$,

$$P(A + B) = P(A) + P(B), \ \ A \cap B = \emptyset$$

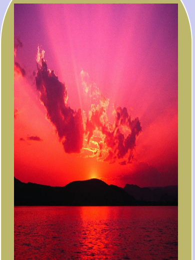
As a generalization of linear expectations and probability measures:

Nonlinear case: $\mathcal{E}[\xi] : \xi \in \mathcal{L} \to R$ nonlinear functional,

or $V : \mathcal{F} \to [0,1]$, nonlinear probability,

$$V(A + B) = V(A) + V(B), \ \ A \cap B = \emptyset$$

is no longer true.



Four modes of nonlinear expectations

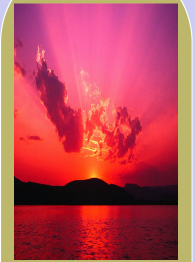
Nonlinear expectations as an alternative to mathematical expectations are being used extensively in robustness, finance and insurance literature.

(1) Choquet expectations (integral)(Choquet 1953). Potential Theory

(2) g-expectations (Peng 1997). Backward stochastic differential equations. Control Theory.

(3) Coherent risk measures (Artzner-Delbaen-Eber-Heath,1999). Asset Pricing Theory.

(4) Convex risk measures (Föllmer and Schied 2002).



1. Choquet expectation

Nonlinear: $V(\cdot) : \mathcal{F} \to [0,1]$ but

$$V(A+B) \neq V(A) + V(B), \text{even if} A \cap B = \emptyset.$$

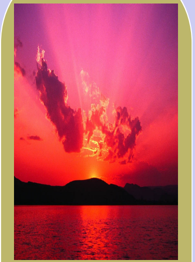
Choquet expectation (1953)

$$C_v(\xi) = \int_{-\infty}^{0} [V(\xi \geq t) - 1]dt + \int_{0}^{\infty} V(\xi \geq t)dt$$

$\star$ Property of Choquet expectation:

$$C_v(\xi + \eta) = C_v(\xi) + C_v(\eta)$$

is no longer true.



Home Page
Title Page
Page 9 of 18
Go Back
Full Screen
Close
Quit

2. BSDE and g-expectations

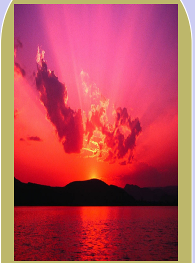
(1)BSDE [Pardoux and Peng, 1990]: $\xi \in L^2(\Omega, \mathcal{F}, P)$, g-Lip continuous,

$$y_t = \xi + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dW_s, \quad t \in [0, T].$$

(2) g-expectation [Peng, 1997]: $g(y, 0, t) = 0, \forall y, t.$

$$\mathcal{E}_g[\xi] = y_0. \quad \mathcal{E}_g[\cdot] : L^2 \to R.$$

Particularly, $\mathcal{E}_g[\xi] = E\xi$ if $g \equiv 0$.



Home Page
Title Page
Page 10 of 18
Go Back
Full Screen
Close
Quit

3. Risk Measures

$\rho : X \to R :$

⋆Coherent risk measures:[Artzner-Delbaen-Eber-Heath,1999]

(1) Super-additivity: for all X_1, X_2, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

(2) Positive homogeneity: for all $\lambda \geq 0$ and all X, $\rho(\lambda X) = \lambda \rho(X)$.

(3) Monotonicity: for all X and Y with $X \geq Y$, $\rho(X) \geq \rho(Y)$.

(4) Translation invariance: for X, $c \in R$, $\rho(X + c) = \rho(X) + c$.

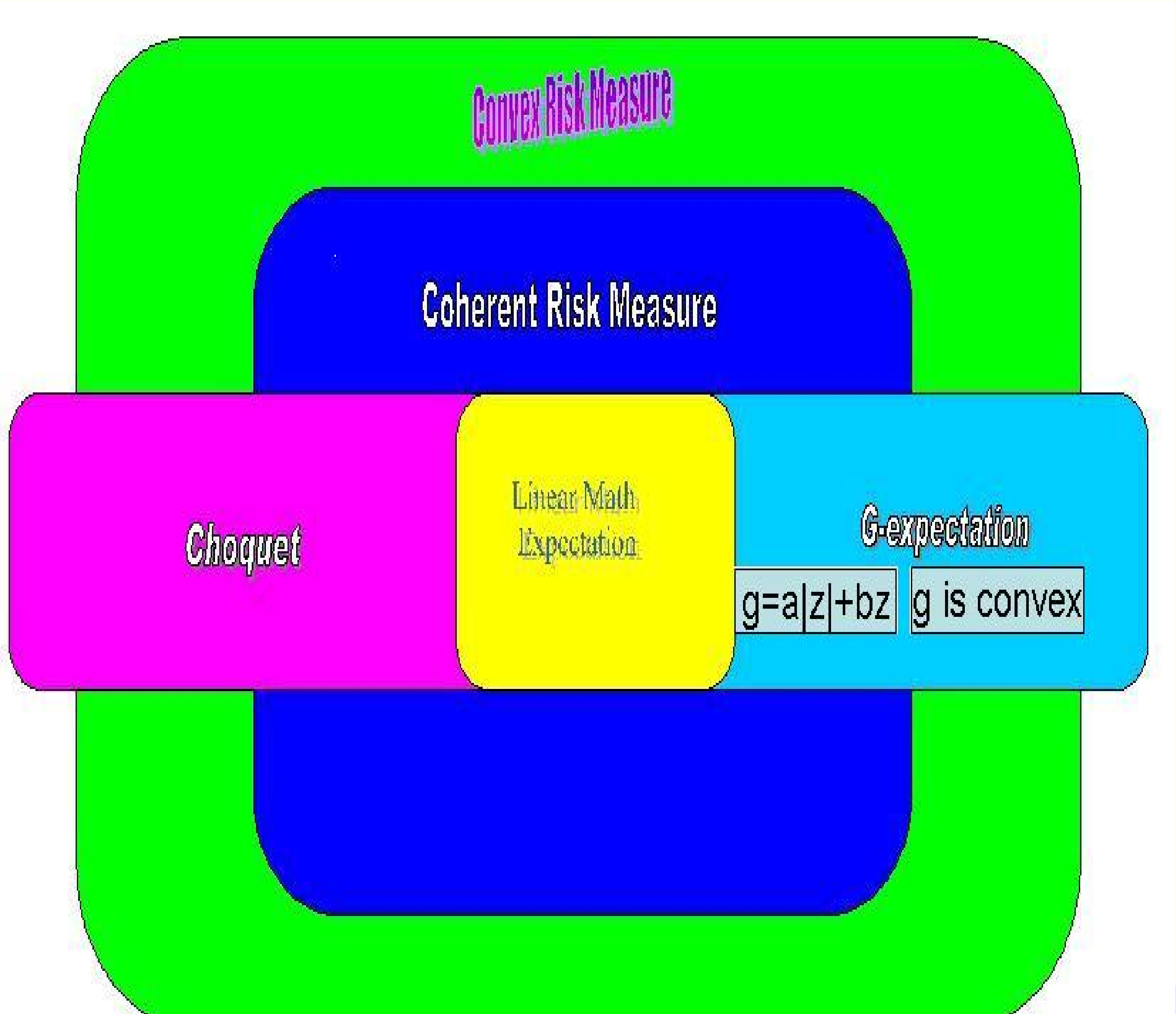
⋆ Convex risk measures:[Föllmer and Schied 2002]:

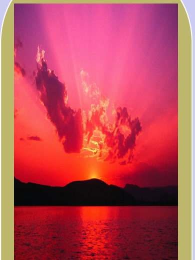
(i) Convexity: $\rho(\lambda X_1 + (1-\lambda) X_2) \leq \lambda \rho(X_1) + (1-\lambda)\rho(X_2), \forall \lambda \in [0,1]$;

(ii) Normality: $\rho(0) = 0$;

(iii) Properties (3) and (4) in coherent risk measures.

4. Relations among nonlinear expectations





5. Black-Scholes Model

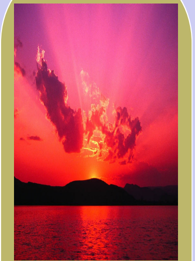
Consider a financial market, d stocks governed by linear SDE, one bond. Value process $V_t :\equiv V_t^{x,\pi}$ (or increment dV_t of V_t) satisfies SDE:

$$dV_t = [rV_t + \pi_t^* \sigma_t \theta]dt + \pi_t^* \sigma_t dW_t, \quad V_0 = x. \tag{1}$$

where $\theta = [b - r\mathbf{1}]\sigma_t^{-1}$.

Pricing Principle: Given a contingent claim ξ at time T, the question of hedging contingent claim ξ in fact is to seek an initial endowment $\hat{x}$ and portfolio process $\hat{\pi}$ such that the market value (wealth) process $\{V_t^{\hat{x},\hat{\pi}}\}$ satisfies $V_T^{\hat{x},\hat{\pi}} = \xi$. The fair price of claim ξ is defined as the minimal endowment $\hat{x}$.

Black-Scholes Theory: there exists E_Q such that $\hat{x} = E_Q\left[\xi e^{-rT}\right]$.



6. Questions

Recently, in studying the pricing of contingent claim with constraint on wealth or portfolio processes, e.g.

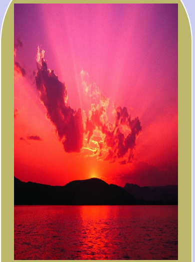
⋆ **higher interest rate for borrowing** (Cvitanic and Karatzas(1993))

$$dV_t = [rV_t + \pi_t^* \sigma_t \theta]dt + \pi_t^* \sigma_t dW_t - (R - r)(V_t - \sum_{i=1}^{d} \pi_t^i)^- dt, \; V_0 = x \quad (2)$$

⋆ Short sales constraint(Jouini and Kallal 1995, He and Pearson 1991):

$$dV_t = [rV_t + \pi_t^* \sigma_t \theta^1]dt + \pi_t^* \sigma_t dW_t + [\pi_t^*]^- \sigma_t [\theta^1 - \theta^2]dt, V_0 = x. \quad (3)$$

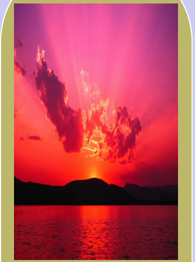
⋆ **Question:** Contrast to linear pricing, could we identify x as nonlinear expectations when the market value process is either SDE(2) or SDE(5)? $\mathcal{E} = ?$, $\hat{x} = \mathcal{E}[\xi e^{-rT}]$. Choquet pricing? g-expectation pricing? Coherent pricing? Convex pricing?



7. Definition

DEFINITION 1 *Given a market value (wealth) process* $\{V_t^{x,\pi}\}$,*if there is a mapping* $\mathcal{E}[\cdot] : L^2 \to R$ *such that for any claim* $\xi \in L^2(\Omega, \mathcal{F}, P)$, *let* $x = \mathcal{E}[e^{-rt}\xi]$, *and a portfolio* π *such that* $V_T^{x,\pi} = \xi$, *we say that the market value process could be priced by* $\mathcal{E}[\cdot]$.

The sub-price of a claim corresponding to $\mathcal{E}[\cdot]$ is still defined as the minimal endowment x, that is $\min\{x | V_T^{x,\pi} = \xi\}$



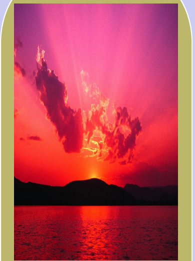
8. Result for higher interest rate for borrowing

THEOREM 1 **Higher interest rate for borrowing model:**

$$dV_t = [rV_t + \pi_t^* \sigma_t \theta]dt + \pi_t^* \sigma_t dW_t - (R - r)(V_t - \sum_{i=1}^{d} \pi_t^i)^- dt, \; V_0 = x \qquad (4)$$

Contingent claims with higher interest rate for borrowing could be priced by g-expectations, but not by convex risk measures (coherent risk measures, Choquet expectations). That is, there exists a g-expectation such that for any claim $\xi \in L^2$, denote $\hat{x} = \mathcal{E}_g[\xi e^{-rT}]$, there exists a portfolio $\hat{\pi}$ such that $V_T^{\hat{x},\hat{\pi}} = \xi$.

g-expectation is the best in this setting!



9. Result for short-sale constaint

Short sales constraint model:

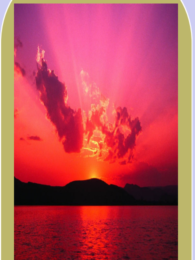
$$dV_t = [rV_t + \pi_t^* \sigma_t \theta^1]dt + \pi_t^* \sigma_t dW_t + [\pi_t^*]^- \sigma_t [\theta^1 - \theta^2]dt, V_0 = x. \qquad (5)$$

THEOREM 2 *(1) Contingent claims with short-sales constraints could be priced by both g-expectations and coherent risk measures, but not by Choquet expectations. Moreover, if the coherent risk measure is ρ, let $V(A) := \rho(I_A)$, then*

$$\rho(\xi) \leq C_V(\xi), \xi \in L^2.$$

(2) However, if contingent claims are European with the form $\xi = (S_T - k)^+$, where S_t is a geometric Brownian motion, then those claims can be priced by a Choquet expectation moreover,

$$\rho(\xi) = C_V(\xi), \xi \in L^2.$$

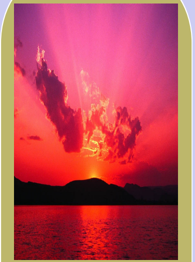


10. Main Result

THEOREM 3 *Given a value process $V_t^{x,\pi}$ (shortly V_t), claims $\xi \in L^2$) can be priced by a g-expectation, iff there exist two Ito type processes X and Y with $E|X_t|^2 < \infty$, $E|Y_t|^2 < \infty$ such that the increments of V_t, X_t and Y_t satisfy*

$$dX_t \leq dV_t \leq dY_t$$

Here g-expectation is more general (See Peng 1999).



Thank you !