HIGH MOMENT ASYMPTOTICS FOR LOCAL AND INTERSECTION LOCAL TIMES

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1. Gärtner-Ellis large deviation

Given a non-negative random variable X and $\gamma > 0$, if there is a $\lambda_o > 0$ such that

$$\mathbb{E}\exp\left\{\lambda X^{1/\gamma}\right\}\begin{cases} < \infty & \lambda < \lambda_o \\ \\ = \infty & \lambda > \lambda_o \end{cases}$$

then "very likely"

$$\lim_{t\to\infty} t^{-1/\gamma}\log \mathbb{P}\{X \geq t\} = -\lambda_o.$$

In the spirit of Taylor expansion, we have

Lemma. *(König and Mörters (2002)) Let $\gamma > 0$ and assume that*

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^\gamma}\mathbb{E}\, X^m = -\kappa.$$

Then

$$\lim_{t\to\infty} t^{-1/\gamma}\log \mathbb{P}\{X \geq t\} = -\gamma e^{\kappa/\gamma}.$$

In this talk, we discuss a newly developed method of computing the asymptotics of

$$\mathbb{E}\, X^m$$

when X is an intersection local time or local time.

2. Intersection in supercritical dimensions

Let $X_1(t), \cdots, X_p(t) \quad (t \geq 0)$ be independent, identically distributed, symmetric and square integrable random walks on $\mathbb{Z}^d$. $p(d-2) > d$.

The subject of interest is the intersection local time

$$
\begin{aligned}
I &= \text{Lebesgue measure}\Big((t_1, \cdots, t_p); \\
&X_1(t_1) = \cdots = X_p(t_p)\Big) \\
&= \int_0^\infty \cdots \int_0^\infty 1_{\{X_1(t_1) = \cdots = X_p(t_p)\}} dt_1 \cdots dt_p \\
&= \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^p \int_0^\infty 1_{\{X_j(t) = x\}} dt
\end{aligned}
$$

which is well defined under $p(d-2) > d$.

$$
J = \#\{X_1[0,\infty) \cap \cdots \cap X_p[0,\infty)\}
$$

We call I intersection local time, and J range intersection.

Quantities measuring the amount of intersection of random walks have been studied intensively for more than twenty years. This research is often motivated by the role these quantities play in renormalization group methods for quantum field theory, in our understanding of polymer models.

In the special case $p = 2$ and $d \geq 5$ of the supercritical dimensions, Khanin, Mazel, Shlosman and Sinai (1994) observed the following surprising behaviors:

$$\exp\{-c_1 t^{1/2}\} \leq \mathbb{P}\{I \geq t\} \leq \exp\{-c_2 t^{1/2}\}$$

$$\exp\{-t^{\frac{d-2}{d}+\epsilon}\} \leq \mathbb{P}\{J \geq t\} \leq \exp\{-t^{\frac{d-2}{d}-\epsilon}\}$$

for large $t > 0$.

Similar observations were made at around the same time (1993) by Sznitman in a continuous setting

The challenging question lies in understanding the difference between I and J, providing sharp estimates for the tails.

We provide a complete solution for I.

Theorem 1. *(Mörters and Chen (2007))*

Under $p(d-2) > d$,

$$\lim_{t\to\infty} \frac{1}{t^{1/p}} \log \mathbb{P}\{I \geq t\} = -\frac{p}{\rho}$$

where

$$\rho = \sup_f \sum_{x,y\in\mathbb{Z}^d} G(x-y)f(x)f(y),$$

$G(x)$ is the Green's function defined by

$$G(x) = \int_0^\infty \mathbb{P}\{X(t) = x\}dt \quad x \in \mathbb{Z}^d$$

and the supremum is taking over the functions f on $\mathbb{Z}^d$ with

$$\sum_{x\in\mathbb{Z}^d} |f(x)|^{\frac{2p}{2p-1}} = 1.$$

The proof is based on the establishment of the high moment asymptote

$$\lim_{m\to\infty} \frac{1}{m} \log \frac{1}{(m!)^p} \mathbb{E}\, I^m = p \log \rho.$$

Indeed, notice that

$$I = \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^{p} \int_0^\infty 1_{\{X_j(t)=x\}} dt.$$

Hence,

$$I^m = \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} \prod_{j=1}^{p} \prod_{k=1}^{m} \int_0^\infty 1_{\{X_j(t)=x_k\}} dt.$$

This gives

$$\mathbb{E}\, I^m = \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} \bigg[\sum_{\sigma \in \Sigma_m} \int_{\{0 \le t_1 \le \cdots \le t_m\}} dt_1 \cdots dt_m$$

$$\times \prod_{k=1}^{m} \mathbb{P}\{X(t_k - t_{k-1}) = x_{\sigma(k)} - x_{\sigma(k-1)}\} \bigg]^p$$

$$= \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} \bigg[\sum_{\sigma \in \Sigma_m} \prod_{k=1}^{m} G(x_{\sigma(k)} - x_{\sigma(k-1)}) \bigg]^p.$$

The problem is reduced to prove

$$\lim_{m \to \infty} \frac{1}{m} \log \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} \bigg[\frac{1}{m!} \sum_{\sigma \in \Sigma_m}$$

$$\times \prod_{k=1}^{m} G(x_{\sigma(k)} - x_{\sigma(k-1)}) \bigg]^p = p \log \rho.$$

The idea of dealing high moment associated with large permutation group first appeared in König and Mörters (2002).
The key of the proof is to get rid of permutations. We only prove the lower bound (the delicate part is the upper bound).

Let $q > 1$ the the conjugate of p and let $f(x) \geq 0$ on $\mathbb{Z}^d$ such that

$$\sum_{x \in \mathbb{Z}^d} f^q(x) = 1$$

By Hölder inequality,

$$\left\{ \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} \left[\sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)}) \right]^p \right\}^{1/p}$$

$$\geq \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} f(x_1) \cdots f(x_m)$$

$$\times \sum_{\sigma \in \Sigma_m} \prod_{k=1}^m G(x_{\sigma(k)} - x_{\sigma(k-1)})$$

$$= m! \sum_{x_1, \cdots, x_m \in \mathbb{Z}^d} f(x_1) \cdots f(x_m) \prod_{k=1}^m G(x_k - x_{k-1})$$

the desired lower bound is associated with the principal eigenvalue problem. □

$$\approx \langle g, T^m g \rangle.$$

Recall that

$$J \equiv \#\{S_1[1,\infty) \cap \cdots \cap S_p[1,\infty)\}$$

The most important recent progress on J was made by van den Berg, Bolthausen and den Hollander (Ann. Math., 2004): To apply the famous Donsker-Varadhan large deviations, they consider the intersection of two independent Brownian sausages $W_1^\epsilon(t)$ and $W_2^\epsilon(t)$ instead of J and show that as $d \geq 5$,

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\Big\{|W_1^\epsilon(\theta t) \cap W_2^\epsilon(\theta t)| \geq t\Big\} = -I_d^\epsilon(\theta)$$

where the rate function $I_d^\epsilon(\cdot)$ is given in the form of variation. They also show that there exists a critical θ^* such that $I_d^\epsilon(\theta) = I_d^\epsilon(\theta^*)$ for all $\theta \geq \theta^*$. This strongly suggests (conjectured in van den Berg *et al*) that

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\Big\{|W_1^\epsilon(\infty) \cap W_2^\epsilon(\infty)| \geq t\Big\} = -I_d^\epsilon(\theta^*).$$

The main difficulty encountered by van den Berg, Bolthausen and den Hollander is that the conventional way fails in dealing with the intersections over an infinite time period.

Our success on I gives new hope in finding the exact tail for the range intersection J. Indeed, our method allows the random walks run up to an infinite time period. We believed that we are on the right track leading to that goal.

Brownian intersection local time

Let $W_1(t), \cdots, W_p(t)$ be independent d-dimensional Brownian motions. According to Dvoretzky, Erdös and Kakutani (1950, 1954), these p independent trajectories intersect if and only if $p(d-2) < d$ (sub-critical dimensions).

In sub-critical dimensions we consider the intersection local time

$$\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)$$
$$=\int_{\mathbb{R}^d}\left[\prod_{j=1}^p\int_0^{t_j}\delta_x\big(W_j(s)\big)ds\right]dx.$$

We are interested in the tail probability of

$$\alpha\big([0,1]^p\big).$$

Let $\tau_1, \cdots, \tau_p$ be independent exponential times with mean 1. We assume independence between

$$(\tau_1, \cdots, \tau_p) \text{ and } (W_1, \cdots, W_p).$$

We have

$$\mathbb{E}\,\alpha\big([0,\tau_1]\times\cdots[0,\tau_p]\big)^m$$

$$=\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p$$

where $G(x)$ is the Green's function given by

$$G(x)=\int_0^\infty e^{-t}p_t(x)dx$$

and $p_t(x)$ is the Brownian density.

By the method of high moment asymptotics,

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\,\alpha\big([0,\tau_1]\times\cdots[0,\tau_p]\big)^m=p\log\rho$$

where

$$\rho=\sup_{||f||_{\frac{2p}{2p-1}}=1}\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x-y)f(x)f(y)dxdy.$$

By a argument of Tauberian type, this gives that

$$\lim_{m\to\infty}\frac{1}{m}\log(m!)^{-\frac{d(p-1)}{2}}\mathbb{E}\left[\alpha\big([0,1]^p\big)^m\right]$$
$$=p\log\rho+\frac{2p-d(p-1)}{2}\log\frac{2p}{2p-d(p-1)}$$

which leads to

Theorem 2. *(Chen (2004)). Under $p(d-2)<d$,*

$$\lim_{t\to\infty}t^{-\frac{2}{d(p-1)}}\log\mathbb{P}\Big\{\alpha\big([0,1]^p\big)\geq t\Big\}$$
$$=-\frac{d(p-1)}{2}\Big(\frac{2p-d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{d(p-1)}}\rho^{-\frac{2p}{d(p-1)}}.$$

Remark. The constant on the right hand side was initially given in terms of the best constant $\kappa(d,p)$ of the Gagliardo-Nirenberg inequality

$$||f||_{2p}\leq C||\nabla f||_2^{\frac{d(p-1)}{2p}}||f||_2^{1-\frac{d(p-1)}{2p}}\qquad f\in W^{1,2}(\mathbb{R}^d).$$

Indeed, one can prove that

$$\rho=\Big(\frac{2p-d(p-1)}{2p}\Big)^{\frac{2p-d(p-1)}{2p}}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p}}\kappa(d,p)^2.$$

4. Local time of additive Brownian motion

Let $W_1(t), \cdots, W_p(t)$ be independent d-dimensional Brownian motions. The multi-parameter process

$$W_1(t_1) + \cdots + W_p(t_p) \quad (t_1, \cdots, t_p) \in (\mathbb{R}+)^p$$

is called additive Brownian motion, which is used to "simulate" Brownian sheet.

The local time

$$\eta\big([0, t_1] \times \cdots \times [0, t_p]\big)$$
$$= \int_0^{t_1} \cdots \int_0^{t_p} \delta_0\big(W_1(s_1) + \cdots + W_p(s_p)\big) ds_1 \cdots ds_p.$$

exists if and only if $d < 2p$.

We are interested in the tail probability of

$$\eta\big([0, 1]^p\big).$$

Let $\tau_1, \cdots, \tau_p$ be independent exponential times with mean 1. We assume independence between

$$(\tau_1, \cdots, \tau_p) \text{ and } (W_1, \cdots, W_p).$$

We have

$\lambda_{\sigma(k)} - \lambda_{\sigma(k-1)}$

$$\mathbb{E}\,\eta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m$$

$$= \frac{1}{(2\pi)^d}\int_{(\mathbb{R}^d)^m} d\lambda_1\cdots d\lambda_m \Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k \lambda_{\sigma(j)}\Big)\Big]^p$$

where

$$Q(\lambda) = \big(1 + 2^{-1}|\lambda|^2\big)^{-1} \quad \lambda\in\mathbb{R}^d$$

is the Fourier transform of $G(x)$.

The method of high moment asymptotics gives

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\,\eta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m$$

$$= \log\frac{\widehat{\rho}}{(2\pi)^d}$$

where

$$\widehat{\rho} = \sup_{||f||_2=1}\int_{\mathbb{R}^d} d\lambda\Big[\int_{\mathbb{R}^d}\sqrt{Q(\lambda+\gamma)Q(\gamma)}f(\lambda+\gamma)f(\gamma)d\gamma\Big]^p.$$

By a Tauberian type argument,

$$\lim_{m\to\infty} \frac{1}{m} \log(m!)^{-d/2}\mathbb{E}\left[\eta([0,1]^p)^m\right]$$
$$= \log\left(\frac{2p}{2p-d}\right)^{\frac{2p-d}{2}} + \log\frac{\widehat{\rho}}{(2\pi)^d}.$$

Consequently, we obtain

Theorem 3. *(Chen (2006)) Under* $d < 2p$,

$$\lim_{t\to\infty} t^{-2/d} \log \mathbb{P}\Big\{\eta\big([0,1]^p\big) \geq t\Big\}$$
$$= -(2\pi)^2 \frac{d}{2}\left(1 - \frac{d}{2p}\right)^{\frac{2p-d}{d}} \hat{\rho}^{-2/d}.$$

Conclusion.

The method of high moment asymptotics opens up new avenues for the treatment of a wide range of intersection and multi-parameter problems, which are by no means limited to random walk and Brownian motion. For example, we have reach the point where we know what to expect for the quantities of the form

$$\int_{\Omega^m} \pi(dx_1)\cdots\pi(dx_m)\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)}, x_{\sigma(k)})\Big]^p$$

The mathematical challenge is compatification and discritization of the state space Ω. The reward of success is complete understand of the intersection tail behaviors of the general Markov processes.

$$\sup_f \int_\Omega K(x,y) f(x) f(y).$$

$$\frac{1}{t}\int_0^t \delta_{Z(s)}\, ds.$$