Nonergodicity of Markov Processes

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1.Introduction

The problem——to decide whether a Markov process has a unique invariant measure or not. By nonergodicity we mean nonuniqueness of invariant measures.

Let E be a Polish space and $\{P_x, x \in E\}$ be a homogeneous Feller Markov process on $\Omega = D([0,\infty], E)$. Let

$M_i(E)$=set of invariant (probability) measures of the M.P.

Then the problem is to determine if $|M_i(E)| > 1$. For certain concrete processes, some reasonable conditions can be formulated for uniqueness of invariant measures. Such conditions generally provide no information on nonergodicity. Conditions for nonergodicity were established mostly for some special processes.

1.Introduction

In this presentation we will describe an approach for detecting nonergodicity via an empirical test, based on a single path. The idea behind such an approach is that: if a process is nonergodic, and we run it for a long time, and divide this time interval into subintervals with moderate length, then it is expected that with high probabiliy we can find two subintervals, in which the process behaves in typically different ways. Such an approach was invented by [Comets 94] to detec phase transition for Gibbs fields. The main technique used is a large deviation analysis. This approach was then adopted by [dai pra 94] to study nonergodicity for spin-flip particle systems, based on their large deviation results on such systems.

However, in the later situation, in order to detect nonergodicity, observations in a large box of sites for a long time are needed. this motivates our investigation discussed in this talk, i.e., we would like to know if we can carry out the detection by using observations in a finte box of sites with a bounded size. We prove that for any attractive spin system, we only need to compute the occupation time on a single site to detect nonergodicity. As a by-product, we obtain an Erdös-Rényi type of law for the occupation time. In the ergodic case, we also prove certain exponential ergodicity.

Let $\{P_x, x \in E\}$ be a Markov process on Ω. Given any bounded continuous function f on E, we can formulate an empirical test for nonergodicity as follows: For properly chosen $l_t \leq t$ for $t > 0$, define for $\omega \in \Omega$ and $n \geq 1$

$$T_n^+ = \max_{k \leq n-l_n} \frac{1}{l_n} \int_k^{k+l_n} f(\omega_s)ds, \quad \text{and} \quad T_n^- = \min_{k \leq n-l_n} \frac{1}{l_n} \int_k^{k+l_n} f(\omega_s)ds.$$

Also, denote

$$m_f^+ = \sup\{E^\nu(f),\ \nu \in M_i(E)\} \quad \text{and} \quad m_f^- = \inf\{E^\nu(f),\ \nu \in M_i(E)\}.$$

Then we have the following

2.The general cases

Theorem 2.1. Under the above notations, if

$$\lim_{n\to\infty} \frac{\log n}{l_n} = 0,$$

then for a properly chosen $\gamma > 0$, for any $\delta > 0$, there is a constant $k_\delta > 0$, such that for sufficiently large n,

$$\sup_x P_x(\max\{d(x,\ [m_f^-,\ m_f^+]),\ x \in [T_n^-,\ T_n^+]\} > \delta) \le e^{-k_\delta \gamma l_n}.$$

An implication of the above theorem is the following

Corollary 2.2. Let l_n be chosen as above. If for some $x \in E$ and some $\delta > 0$,

$$\limsup_{n\to\infty}(T_n^+ - T_n^-) \geq \delta,\ P_x - a.s.,$$

then the process is nonergodic.

3.The case of IPS–An Erdös-Rényi law

Now we consider a typical class of IPS-spin flip systems. The Markov process used to modle such a system is driven by a family of bounded local dynamics

$$\{c(i,\ x),\ i \in Z^d,\ x \in E = \{-1, 1\}^{Z^d}\}$$

. We assume that the system is translation invariant with finite range interactions, i.e., we assume

$$c(i,\ x) = c(0,\ \theta_i x),$$

and $c(0,\ x)$ depends on x only through $\{x(i),\ i \in \Lambda\}$ for some fixed finite subset Λ of Z^d. We also assume that the system is attractive(see [Liggett 85] for definition). Then it is known that there is a lower invariant measure ν_-, and an upper invariant measure ν_+ of the system, such that in certain sense

$$\nu_- \leq \nu \leq \nu_+,\ \forall \nu \in M_i(E)$$

and that $|M_i(E)| = 1$ iff $\nu_- = \nu_+$. In such cases, Theorem 2.1 can be refined to the following E

Theorem 3.1 Under the above assumptions and notations, we can choose $\gamma > 0$, such that with $l_n = (\gamma \log n)^{1/d}$, for each $x \in E$,

$$\min_{k \le n - l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds \to \rho_- = E^{\nu_-} \omega(0) \; P_x - a.s.,$$

and

$$\max_{k \le n - l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds \to \rho+ = E^{\nu_+} \omega(0) \; P_x - a.s..$$

Inparticular, the process is nonergodic iff for some $\lambda > 0$,

$$\max_{k \le n - l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds - \min_{k \le n - l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds \to \lambda, \; P_x - a.s.$$

for each $x \in E$.

This suggests that

$$T_n = \max_{k \leq n-l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds - \min_{k \leq n-l_n} \frac{1}{l_n} \int_k^{k+l_n} \omega_s(0) ds$$

is a good candidate for detecting nonergodicity.

Theorem 3.2. For each $\lambda \in [0,\ \rho_-)$, there is a $\gamma_\lambda > 0$, such that

$$\sup_x P_x(\frac{1}{t}\int_0^t \omega_s(0) ds \leq \lambda) \leq e^{-\gamma_\lambda t}\ \forall t > 0.$$

Similarly, for each $\lambda \in (\rho_+, 1]$, there is a $\gamma_\lambda > 0$, such that

$$\sup_x P_x(\frac{1}{t}\int_0^t \omega_s(0) ds \geq \lambda) \leq e^{-\gamma_\lambda t}\ \forall t > 0.$$

In particular, if the process is ergodic with $\rho_- = \rho_+ = \rho$, Then for any $\lambda > 0$, there is a $\gamma_\lambda > 0$, such that

$$\sup_x P_x(|\frac{1}{t}\int_0^t \omega_s(0)ds - \rho| \geq \lambda) \leq e^{-\gamma_\lambda t}\ \forall t > 0.$$

Thanks!