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# The Contact Process beyond $Z^d$

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# Contact Process $\leftrightarrow$ Oriented Percolation

Percolation beyond $Z^d$: many questions and a few answers

I. Benjamini & O. Schramm, Elect Comm in Probab. 1, (1996) 71-82.

The contact process is a Markov process defined on $\{0, 1\}^V$, where $V$ is the vertices set of graph $G = (V, E)$.

Configuration $\eta = \{\eta(x), x \in V\}$

$\eta(x) = 0$ the person at site $x$ is healthy;

$\eta(x) = 1$ the person at site $x$ is infected.

The process can be formally defined by specifying the infinitesimal rates of the generator.

$\mathbf{1 \longrightarrow 0}$ at rate 1

$\mathbf{0 \longrightarrow 1}$ at rate $\boldsymbol{\lambda} \times |\{\boldsymbol{y \sim x; \eta_t(y) = 1}\}|$

$\boldsymbol{\lambda}$ is the only parameter of the process.

Graphic Construction by T. Harris

Infection persists if $\lambda$ is large, and dies out if $\lambda$ is small.

There is a phase transition and a critical value $\lambda_c$,

$0 < \lambda_c < \infty$.

The subcritical Case: $\lambda < \lambda_c$, $\delta_0$ is the only invariant measure (ergodic)

The supercritical Case: $\lambda > \lambda_c$, $\exists$ a family of invariant measures,
In some sense, $\delta_0$ is the ”minimum”, $\delta_1$ is the ”maximum”.
(non-ergodic).

Furthermore, the complete convergence theorem holds for $G = Z^d$.

$$\mu = \alpha\delta_1 + (1 - \alpha)\delta_0.$$

New Phenomenon:

The contact process on $T_d$ has a second phase transition!

*The contact process on trees,* R. Pemantle, Ann. Probab., 20,(1992) 2089-2116.

Liggett (1996)(tour de force), Stacy(1996) survival $\longrightarrow$ weak survival + strong survival,

Two critical values: $\boldsymbol{\lambda_1 < \lambda_2}$.

$\boldsymbol{\lambda < \lambda_1}$, the contact process eventually dies out.

$\boldsymbol{\lambda > \lambda_2}$, strong survival. the infection persists, every site is infected infinitely many times, and the the complete convergence theorem holds.

$\lambda_1 < \lambda < \lambda_2$, weak survival. The infection persists.

However, every site is infected only finite times. (Infection moves away to infinity.)

The complete convergence theorem fails.

$\exists$ infinitely many non-homogeneous invariant measures.

Each boundary condition corresponds an invariant measure.

$\exists$ an invariant measure which is rotationally symmetric.

Ref. the second book of Liggett

**Conjecture:** $\lambda_1 < \lambda_2 \iff$ the Cheeger constant of $G$ is positive.

$S \subset V(G)$, $|S|$ the cardinality of $S$.

$\partial S$ the set of boundary edges.

**Cheeger constant**

$$\iota(G) = \inf \frac{|\partial S|}{|S|}$$

the infinium is over all finite connected subsets.

The special case $\boldsymbol{G = T_d \times Z}$.

**Theorem** (Shumei Jia); If $\boldsymbol{d \geq 8}$, then $\boldsymbol{\lambda_1 < \lambda_2}$.

**Open Problem 1**: Extend the statement to the case $\boldsymbol{d = 3, 4, 5, 6, 7}$.

**Theorem** (Shumei Jia); The complete convergence theorem holds for $\boldsymbol{\lambda > \lambda_2}$ and fails for $\boldsymbol{\lambda_1 < \lambda < \lambda_2}$.

**Open Problems 2**: Find out all the invariant measures.

## Growth Profile, the Shape Theorem

$A_t = \{x \in V, \eta_s(x) = 1$ for some $s \leq t\}$, the set of vertices that ever be infected by time $t$.

For $Z^d$, there is a convex set B such that

$$B(1-\epsilon)t \subset A_t \subset B(1+\epsilon)t.$$

For $T_d$, there is a function $\phi$ which decays exponentially.

$$P_0(\eta_t(x) = 1 \text{ for some} t > 0) = \phi(|x|).$$

**Open Problem 3**: Is this true for $T_d \times Z$? Apparently, the exponential growth of trees make the parallel argument impractical.

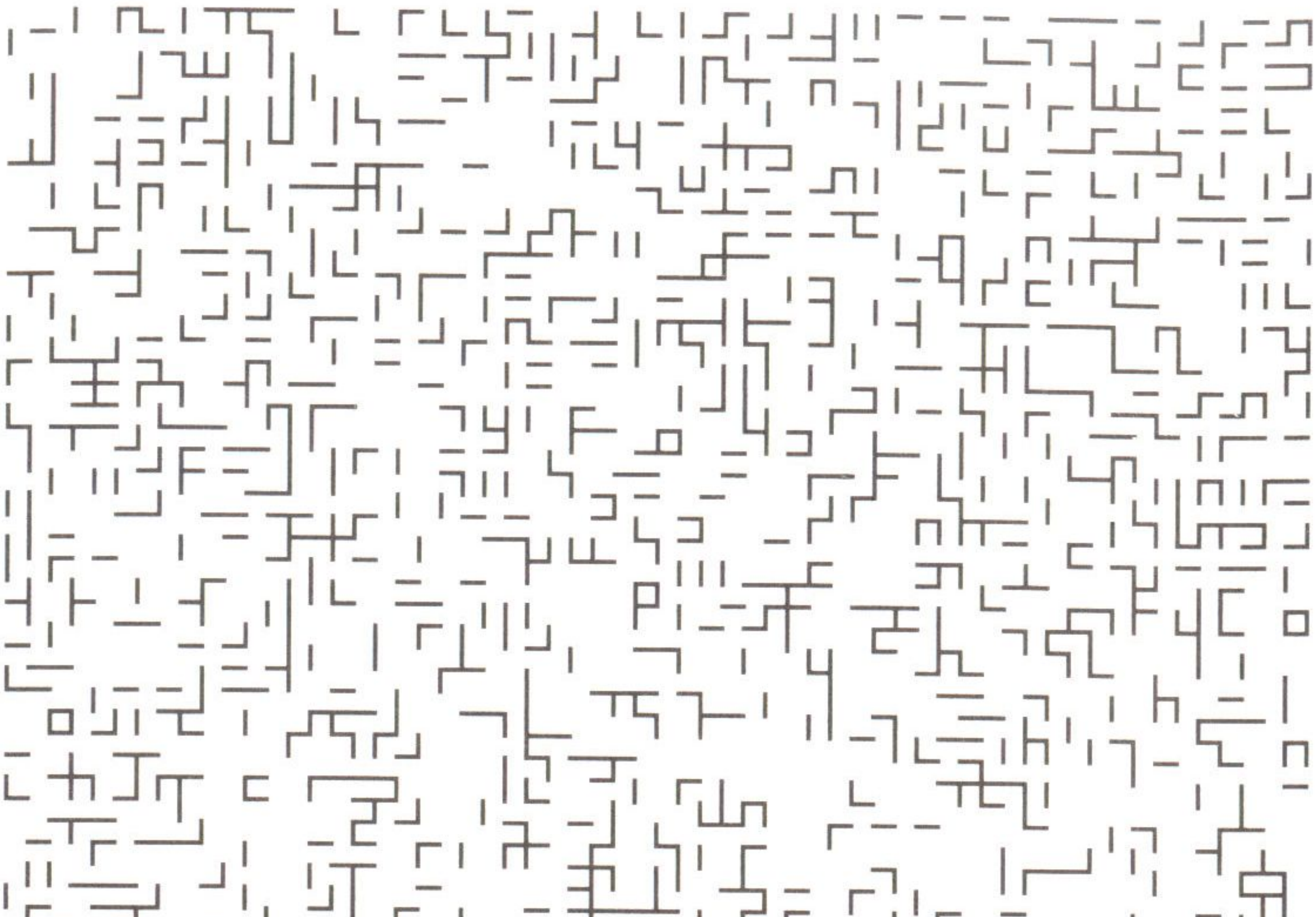
Other transitive graphs?

Percolation. A random environment is the second best.

Let $G = (V, E)$ be an infinite graph. Each edge of $G$ is independently declared *open* with probability $p$ and *closed* with probability $1 - p$.

All open bonds, together with all the vertices, consist of a subgraph. A connected component is called an open cluster.

G.R. Grimmett, "Percolation", 2nd ed. Springer-Verlag, New York, 1999.



The **open cluster** containing the fixed vertex $o$

$$\mathcal{C} = \{x \in V; x \text{ and } o \text{ are connected by open edges}\}.$$

$P(\mathcal{C}$ is infinite$)$ is increasing in $p$.

**Critical value** $p_c = \inf\{p, P(\mathcal{C} \text{ is infinite}) > 0\}$.

The critical probability of bond percolation on the square lattice equals 1/2,

H. Kesten, Comm. Math. Phys. 74, (1980) 41-59.

Suppose that $p > p_c$ and that $\mathcal{C}$ is infinite.

Define the contact process on $\mathcal{C}$.

If $G = T_d$, then $\mathcal{C}$ is a Galton-Watson tree.

**Theorem** (C. & Zhao) If $p > (2\sqrt{d} + 1)/d$, then $\lambda_1 < \lambda_2$.

Remarks: 1. $p_c = 1/d$;

2. Useful only if $d \geq 6$.

Open Problem 4: Find a better condition for $\lambda_1 < \lambda_2$.

Let $G = Z^d$, $\mathcal{C}$ is an infinite cluster.

Question 5. Is $p_c(\mathcal{C}) > p_c(Z^d)$?

Question 6. Is the complete convergence theorem holds for $p > p_c(\mathcal{C})$?

Question 7. Does the infected area grows linearly on the cluster of $Z^d$ and $T_d$?

Thank You

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