Decay Properties of Stopped Markovian Queues with Batch Arrivals

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1. Introduction and Background

Consider Markovian Queue with Batch Arrivals that stops after hitting state 0, or the stopped $M^X/M/1$ queue (crucial in realizing busy period behavior of the queuing systems). The generator matrix,the q-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$, here $\mathbf{Z}_+$ stands for the nonnegative integers $\{0, 1, 2, \cdots\}$ is given by:

$$q_{ij} = \begin{cases} b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

where

$$b_j \geq 0 \ (j \neq 1), \ 0 < \sum_{j \neq 1} b_j \leq -b_1 < \infty. \tag{1.2}$$

Assume that $b_0 > 0$ and $\sum_{j=2}^{\infty} b_j > 0$ and thus $C = \{1, 2, \cdots\}$ is an irreducible class for Q and, also, for the corresponding Q-process.

Definition 1.1. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a generator matrix as abovew. The corresponding transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ is called the stopped $M^X/M/1$ queueing process.

Note that $\sum_{j \neq 1} b_j \leq -b_1$ is allowed.

Since 0 is an absorbing state and $C = \{1, 2, \cdots\}$ is an irreducible but TRANSIENT class, $p_{ij}(t) \to 0$ as $t \to \infty$ for all $i, j \in C$. Hence this process does not possess any limiting distribution in the normal sense. We therefore turn our attention to the decay parameter and quasi-stationary distribution.

By Kingman [9], for an irreducible class C of any CTMC there exists a number $\lambda_C \geq 0$, called the decay parameter of the corresponding process such that for all $i, j \in C$,

$$\frac{1}{t} \log p_{ij}(t) \to -\lambda_C \quad \text{as } t \to +\infty.$$

Let

$$\mu_{ij} = \inf\{\lambda \geq 0 : \int_0^{\infty} e^{\lambda t} p_{ij}(t) dt = \infty\}.$$

Then μ_{ij} does not depend on $i, j \in C$ and the common value of μ_{ij}, denoted by μ, is just the decay parameter, i.e.,

$$\lambda_C = \mu.$$

The decay parameter and quasi-stationary distributions are closely linked with the μ-subinvariant/invariant measures and μ-subinvariant/invariant vectors.

Definition 1.2. Assume that $\mu \geq 0$. A set $(m_i; i \in C)$ of strictly positive numbers is called a μ-subinvariant measure for Q on C if

$$\sum_{i \in C} m_i q_{ij} \leq -\mu m_j, \qquad j \in C. \tag{1.3}$$

If equality holds in (1.3), then $(m_i; i \in C)$ is called a μ-invariant measure for Q on C.

Definition 1.3. Assume that $\mu \geq 0$. A set $(m_i; i \in C)$ of strictly positive numbers is called a μ-subinvariant measure for $p_{ij}(t)$ on C if

$$\sum_{i\in C} m_i p_{ij}(t) \leq e^{-\mu t} m_j, \qquad j \in C. \tag{1.4}$$

If equality holds in (1.4), then $(m_i; i \in C)$ is called a μ-invariant measure for $p_{ij}(t)$ on C.

The subinvariant/invariant vectors can be similarly defined.

Definition 1.4. Suppose that $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ is a Q-process. Assume that C is a communicating class of $\mathbf{Z}_+$ and $(m_i; i \in C)$ is a probability distribution over C. Let $p_j(t) = \sum_{i\in C} m_i p_{ij}(t)$, for $j \in C$ and $t \geq 0$. If

$$\frac{p_j(t)}{\sum_{i\in C} p_i(t)} = m_j, \qquad j \in C, t > 0, \tag{1.5}$$

then $(m_i; i \in C)$ is called a quasi-stationary distribution (qsd).

2. Decay Parameter

For the stopped $M^X/M/1$ queue, define $B(s)$ as the generating function of $\{b_k; k \geq 0\}$, i.e.,

$$B(s) = \sum_{k=0}^{\infty} b_k s^k.$$

Denote the convergence radius of $B(s)$ as

$$\rho = 1/\limsup_{n\to\infty} \sqrt[n]{b_n}.$$

Clearly, $\rho \geq 1$. Now, let

$$\rho_0 = \sup\{s > 0 : B(s) \leq 0\}.$$

Lemma 2.1. *$B(s)$ is convex in $[0, \rho)$ and therefore has either one or two positive roots, denoted by q_S and q_L.*

Remark 2.1. Easy to see that both ρ_0 and $B(\rho_0)$ are finite and there are only two possibilities for ρ_0,i.e. either $\rho_0 = q_L$ or $\rho_0 = \rho < +\infty$.

Now define

$$\lambda^* = \sup\{\lambda \in \mathbf{R} : B(s) + \lambda s = 0 \text{ has a root in } [0, \rho_0]\} \tag{2.1}$$

where $\mathbf{R}$ denotes the set of real numbers. Since $B(0) = b_0 > 0 \geq B(1)$, it is easily seen that

$$\begin{aligned} \lambda^* &= \sup\{\lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } [0, \rho_0]\} \\ &= \sup\{\lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } [q_S, \rho_0]\}. \end{aligned} \tag{2.2}$$

Now further define

$$\bar{\lambda} = \max\{-\frac{B(s)}{s}; s \in [q_S, \rho_0]\}.$$

Since $-\frac{B(s)}{s}$ is a continuous function on the closed interval $[q_S, \rho_0]$ (except the trivial case $q_S = \rho_0$) we know that $\bar{\lambda}$ is finite.

Lemma 2.2. *We have that $\lambda^* = \bar{\lambda}$ and thus $\lambda^* < \infty$ and that the supreme in* (2.1) (*or* (2.2)) *is attainable. In particular, the equation $B(s)+\lambda^* s = 0$ has a unique root $s_* \in [q_S, \rho_0]$ where q_S is given in Lemma 2.1.*

The following is our first main result.

Theorem 2.1. *For the stopped $M^X/M/1$ queue with given Q, $\lambda_C = \lambda^*$ where $C = \{1, 2, \cdots\}$.*

The effective way in obtaining the exact value of the decay parameter λ_C is the following:

Corollary 2.1. *Let $g(s) = B(s) - sB'(s)$.*

(i) *If $g(\rho_0) \geq 0$, then $\lambda_C = -\frac{B(\rho_0)}{\rho_0}$.*

(ii) *If $g(\rho_0) < 0$, then $g(s) = 0$ has a unique root $s_* \in [0, \rho_0)$ and then $\lambda_C = -B'(s_*)$. That is that s_* and λ_C satisfy the equation*

$$\begin{cases} B(s_*) - s_* B'(s_*) = 0 \\ \lambda_C = -B'(s_*) \end{cases} \tag{2.3}$$

or, equivalently, s_* and λ_C are the unique solution of the following equation regarding the unknowns s and λ

$$\begin{cases} B(s) - sB'(s) = 0 \\ \lambda = -B'(s) \end{cases} \tag{2.4}$$

In applying Corollary 2.1, we need to check $g(\rho_0)$ which may not be always convenient since ρ_0 may equal the largest root q_L of $B(s) = 0$. Fortunately, this difficulty can be avoided. In fact, we do not need to find q_L as the following conclusion shows. Note that in the following corollary all the conditions are imposed to the easily obtained quality ρ rather than ρ_0. Here we shall (WLG)only be concerned with the conservative case.

Corollary 2.2. *Let Q be a conservative generator matrix as defined in* (1.1)–(1.2) *and ρ be the convergence radius of $B(s)$.*

(i) *If $B'(1) < 0$ and $\rho = 1$ or if $B'(1) = 0$, then $\lambda_C = 0$ and $s_* = 1$.*

(ii) *If $B'(1) > 0$ (including $B'(1) = +\infty$), then $0 < s_* < 1$ and $\lambda_C > 0$. Moreover, s_* and λ_C can be determined by solving either Equation* (2.4) *or Equation* (2.3) *directly.*

(iii) *If $B'(1) < 0$ and $\rho > 1$ (including $\rho = +\infty$), then $s_* > 1$ and $\lambda_C > 0$. Moreover, if $\rho < +\infty$, $B'(\rho) < 0$ and $B(\rho) > \rho B'(\rho)$, then $s_* = \rho$ and $\lambda_C = -\frac{B(\rho)}{\rho}$ while if any one of the above conditions fails, then s_* and λ_C can be determined by solving either Equation* (2.4) *or Equation* (2.3) *directly.*

Remark 2.2. Corollary 2.2 tells us that only in the case $B'(1) < 0$ and $\rho > 1$, do we need to check whether the condition $B(\rho) > \rho B'(\rho)$ is satisfied or not. Furthermore, even if for the case (iii) in Corollary 2.2, if $\rho = +\infty$ or even if $\rho < +\infty$ but with $B(\rho) \geq 0$ or $B'(\rho) \geq 0$, then we may immediately claim that λ_C and s_* can be determined by solving Equations (2.4) or Equations (2.3) directly. Note further that even if for this last case, we do not need to calculate the exact values $B(\rho)$ and $B'(\rho)$, since what we only need to know is whether $B(\rho) > \rho B'(\rho)$ or not. Later, we shall use several examples to show that how easily this corollary can be applied to find the decay parameter of the corresponding models.

The following corollary shows that the decay parameter has a very clear geometric interpretation.

Corollary 2.3. *Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the stopped $M^X/M/1$ queueing process with generator matrix Q as defined in* (1.1)–(1.2).

(i) *If $-\frac{B(\rho_0)}{\rho_0} < \max\{-\frac{B(s)}{s} : s \in [q_S, \rho_0]\}$, then $y = -\lambda_C x$ is the tangent line of the curve $y = B(x)$;*

(ii) *If $-\frac{B(\rho_0)}{\rho_0} = \max\{-\frac{B(s)}{s} : s \in [q_S, \rho_0]\}$, then $\lambda_C = -\frac{B(\rho_0)}{\rho_0}$.*

3. λ_C-transience Property

We are now interested in realizing whether the stopped $M^X/M/1$ queue process is λ_C-transient or λ_C-recurrent and some other related properties. From now on, we shall always assume that the generator matrix Q is conservative.

Theorem 3.1. *Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the stopped $M^X/M/1$ queue with generator matrix Q. Then for any $\lambda \in (-\infty, \lambda_C]$ and $i \geq 1$*

$$\int_0^\infty e^{\lambda t} p'_{i0}(t) dt = s_\lambda^i, \tag{3.1}$$

and

$$\sum_{j=1}^\infty \left(\int_0^\infty e^{\lambda t} p_{ij}(t) dt\right) \cdot s^{j-1} = \frac{s_\lambda^i - s^i}{B(s) + \lambda s}, \quad |s| < s_\lambda \tag{3.2}$$

where s_λ is the smallest positive root of $B(s) + \lambda s = 0$. Moreover,

$$\int_0^\infty e^{\lambda t} p_{ij}(t) dt = s_\lambda^{i+1-j} \cdot \sum_{k=0}^{(j-1)\wedge(i-1)} \frac{G_\lambda^{(j-k-1)}(0)}{(j-k-1)!}, \quad j \geq 1, \tag{3.3}$$

where $G_\lambda^{(k)}(0)$ denotes the k'th degree derivative of $G_\lambda(s) = \frac{1-s}{B(s_\lambda s) + \lambda s_\lambda s}$ evaluated at 0. In particular, the stopped $M^X/M/1$ queue process is always λ_C-transient.

4. Quasi-stationary Distributions

We now turn our attention to the quasi-stationary distribution. First consider the invariant measures.

Theorem 4.1. *Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Q-function of the stopped $M^X/M/1$ queue with the decay parameter λ_C. Then for any $\lambda \in [0, \lambda_C]$,*

(i) *there exists a λ-invariant measure $(m_i; i \in C)$ for Q on C, which is unique up to constant multiples. Moreover, the generating function of this λ-invariant measure $M(s) = \sum_{i=1}^{\infty} m_i s^{i-1}$ takes the simple form as*

$$M(s) = \frac{m_1 b_0}{B(s) + \lambda s}, \quad |s| < s_\lambda \tag{4.1}$$

where s_λ is the smallest positive root of $B(s) + \lambda s = 0$ and $m_1 > 0$ is a constant.

(ii) *This measure $(m_i; i \in C)$ is also a λ-invariant measure for $(p_{ij}(t); t \geq 0)$ on C.*

(iii) *This λ-invariant measure is convergent (i.e., $\sum_{i \in C} m_i < \infty$) if and only if $B'(1) < 0$, $\rho > 1$ (including $\rho = +\infty$) and $0 < \lambda \leq \lambda_C$, where ρ is the convergence radius of $B(s)$.*

Remark 4.1. Since a λ-invariant measure for $p_{ij}(t)$ on C must be a λ-invariant measure for Q on C. Theorem 4.1 implies that the λ-invariant measure for $p_{ij}(t)$ on C is unique up to constant multiples.

We now further consider the quasi-stationary distributions for $p_{ij}(t)$ on C.

Theorem 4.2. *There exists a qsd for $p_{ij}(t)$ on C if and only if $B'(1) < 0$ and $\rho > 1$. Moreover, if these conditions hold, then there exist one family of quasi-stationary distributions $\{(m_i(\lambda); i \in C); \lambda \in (0, \lambda_C]\}$ which can be given by*

$$M_\lambda(s) = \frac{\lambda}{B(s) + \lambda s}, \quad |s| < s_\lambda \tag{4.2}$$

where $M_\lambda(s) = \sum_{i=1}^{\infty} m_i(\lambda) s^{i-1}$ and s_λ is the smallest positive root of $B(s) + \lambda s = 0$.

The following corollary shows that the λ_C-quasi-stationary distribution has some minimal properties among the family of quasi-stationary distributions.

Corollary 4.1. *Let $\{(m_i(\lambda), i \in C); \lambda \in (0, \lambda_C]\}$ be the one-parameter family of quasi-stationary distributions specified in Theorem 4.2 and let X_λ ($\lambda \in (0, \lambda_C]$) be the corresponding random variable which obeys the distribution $(m_i(\lambda), i \in C)$. Then the λ_C-quasi-stationary distribution $(m_i(\lambda_C); i \in C)$ is the minimal one in the sense that its corresponding random variable X_{λ_C} has the smallest mean value and the smallest variance. Moreover, for any $\lambda \in (0, \lambda_C]$,*

$$E[X_\lambda] = \sum_{i=1}^{\infty} i m_i(\lambda) = -\frac{B'(1)}{\lambda} \tag{4.3}$$

and

$$Var(X_\lambda) = \sum_{i=1}^{\infty} m_i(\lambda)(i + \frac{B'(1)}{\lambda})^2 = \frac{B'(1)^2}{\lambda^2} - \frac{B''(1) - B'(1)}{\lambda}. \tag{4.4}$$

5. Examples

Example 5.1. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be the stopped $M/M/1$ generator matrix defined as:

$$q_{ij} = \begin{cases} b, & \text{if } i \geq 1, j = i+1 \\ a, & \text{if } i \geq 1, j = i-1 \\ -(a+b), & \text{if } i = j \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

where $a > 0, b > 0$. For this example, $\rho = +\infty$ and $B(s) = a - (a+b)s + bs^2$ and thus by Corollary 2.2, we need to solve

$$\begin{cases} a - (a+b-\lambda_C)s + bs^2 = 0 \\ -(a+b) + 2bs = -\lambda_C. \end{cases} \tag{5.2}$$

Solving Equations (5.2) easily yields that

$$\lambda_C = (\sqrt{a} - \sqrt{b})^2, \qquad s_* = \sqrt{a/b}.$$

For any $\lambda \in [0, (\sqrt{a} - \sqrt{b})^2]$, the λ-invariant measure $(m_i(\lambda); i \in C)$ for Q (or for $p_{ij}(t)$) on C is given by

$$M_\lambda(s) = \frac{m_1 a}{a - (a+b-\lambda)s + bs^2}, \qquad |s| < \frac{a+b-\lambda - \sqrt{(a+b-\lambda)^2 - 4ab}}{2b}, \tag{5.3}$$

where $M_\lambda(s) = \sum_{i=1}^{\infty} m_i(\lambda) s^{i-1}$. In particular, a λ_C-invariant measure $(m_i; i \in C)$ for Q (or for $p_{ij}(t)$) on C is

$$m_1 > 0, \quad m_i = i(\sqrt{b/a}\ )^{i-1} m_1, \quad i > 1. \tag{5.4}$$

Moreover, there exists a quasi-stationary distribution for $p_{ij}(t)$ on C if and only if $a > b$. Under this condition, the one-parameter family of quasi-stationary distributions $\{(m_i(\lambda); i \in C); \lambda \in (0, (\sqrt{a} - \sqrt{b})^2]\}$ is given by (5.3) with $m_1 = \lambda/a$. For any $\lambda \in (0, (\sqrt{a} - \sqrt{b})^2]$, we have

$$E[X_\lambda] = \sum_{i=1}^{\infty} i m_i(\lambda) = \frac{a-b}{\lambda} \downarrow \frac{a-b}{(\sqrt{a} - \sqrt{b})^2} \quad \text{as } \lambda \uparrow (\sqrt{a} - \sqrt{b})^2$$

and

$$Var(X_\lambda) = \frac{(a-b)^2 - \lambda(a+b)}{\lambda^2} \downarrow \frac{2\sqrt{ab}}{(\sqrt{a} - \sqrt{b})^2} \quad \text{as } \lambda \uparrow (\sqrt{a} - \sqrt{b})^2$$

In particular, one of the quasi-stationary distributions is

$$m_i = i(1 - \sqrt{b/a})^2 (\sqrt{b/a}\ )^{i-1}, \quad i > 1.$$

Furthermore, for any $i, j \geq 1$, we have

$$\int_0^\infty e^{\lambda_C t} p'_{i0}(t)dt = (\frac{a}{b})^{i/2}$$

and

$$\int_0^\infty e^{\lambda_C t} p_{ij}(t)dt = \frac{j \wedge i}{a} \cdot (\frac{b}{a})^{(j-i-1)/2}.$$

The following example is a generalization of Example 5.1.

Example 5.2. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a stopped $M^X/M/1$ generator matrix defined as follows:

$$q_{ij} = \begin{cases} b, & \text{if } i \geq 1, j = i + k \\ a, & \text{if } i \geq 1, j = i - 1 \\ -(a+b), & \text{if } i = j \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.5}$$

where $a > 0, b > 0, k \geq 1$. The corresponding Q-function is denoted by $(p_{ij}(t); i, j \in \mathbf{Z}_+)$. For this example, we still have $\rho = +\infty$ and $B(s) = a - (a+b)s + bs^{k+1}$ thus we need to solve

$$\begin{cases} a - (a + b - \lambda_C)s + bs^{k+1} = 0 \\ -(a+b) + (k+1)bs = -\lambda_C. \end{cases} \tag{5.6}$$

Solving Equations (5.6) yields that

$$\lambda_C = a + b - (k+1)b[a/(kb)]^{k/(k+1)}, \qquad s_* = \sqrt[k+1]{a/(kb)}.$$

For any $\lambda \in [0, \lambda_C]$, a λ-invariant measure $(m_i(\lambda); i \in C)$ for Q (or for $p_{ij}(t)$) on C can be expressed as

$$M_\lambda(s) = \frac{m_1 a}{a - (a + b - \lambda)s + bs^{k+1}}, \qquad |s| < s_\lambda, \tag{5.7}$$

where $M_\lambda(s) = \sum_{i=1}^\infty m_i(\lambda)s^{i-1}$ and s_λ is the smallest positive root of $a - (a+b-\lambda)s + bs^{k+1} = 0$. By Theorem 4.2, there exists a quasi-stationary distribution for $p_{ij}(t)$ on C if and only if $a > kb$. Under this condition, the one-parameter family of quasi-stationary distributions $\{(m_i(\lambda); i \in C); \lambda \in (0, \lambda_C]\}$ is given by (5.7) with $m_1 = \lambda/a$. Moreover, for any $\lambda \in (0, a + b - (k+1)b \cdot (\frac{a}{kb})^{k/(k+1)}]$, we have

$$E[X_\lambda] = \sum_{i=1}^\infty i m_i(\lambda) = \frac{a - kb}{\lambda} \downarrow \frac{a - kb}{a + b - (k+1)b \cdot (\frac{a}{kb})^{k/(k+1)}} \quad \text{as } \lambda \uparrow \lambda_C$$

and

$$Var(X_\lambda) = \frac{(a-kb)^2 - \lambda(a + k^2 b)}{\lambda^2}.$$

Now we give another example in which the convergence radius of $B(s)$ is finite.

Example 5.3. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a stopped $M^X/M/1$ generator matrix defined as follows:

$$q_{ij} = \begin{cases} \frac{b\theta^{j-i-1}}{(j-i+1)(j-i)}, & \text{if } i \geq 1, j > i \\ a, & \text{if } i \geq 1, j = i-1 \\ -[a + b(\frac{1}{\theta} + \frac{1-\theta}{\theta^2}\ln(1-\theta))], & \text{if } i = j \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.8}$$

where $a > 0, b > 0, \theta \in (0, 1]$. Note that if $\theta = 1$, then as a convention we view $(1-\theta)\ln(1-\theta) = 0$ in (5.8) and some similar expressions below. It is easy to see that the convergence radius of $B(s)$ is $1/\theta$ and when $\theta \in (0, 1)$,

$$B(s) = a - [a + \frac{b(1-\theta)}{\theta^2}\ln(1-\theta)]s + \frac{b(1-\theta s)\ln(1-\theta s)}{\theta^2}, \qquad |s| \leq 1/\theta.$$

For this example, there are two different situations. First, if $\theta < 1$ then $B(s) - B'(s)s = [a\theta^2 + b(\theta s + \ln(1-\theta s))]/\theta^2 = 0$ has a unique root $s_* \in (0, 1/\theta)$ and the decay parameter λ_C of $C = \{1, 2, \cdots\}$ is

$$\lambda_C = -B'(s_*) = a + \frac{b}{\theta} + b[\frac{\ln(1-\theta s_*)}{\theta} + \frac{(1-\theta)\ln(1-\theta)}{\theta^2}].$$

For any $\lambda \in [0, \lambda_C]$, the λ-invariant measure $(m_i(\lambda); i \in C)$ is given by

$$M_\lambda(s) = \frac{m_1 a}{B(s) + \lambda s}, \qquad |s| < s_\lambda, \tag{5.9}$$

where $M_\lambda(s) = \sum_{i=1}^\infty m_i(\lambda)s^{i-1}$ and s_λ is the smallest positive root of $B(s) + \lambda s = 0$. Also, there exists a qsd for $p_{ij}(t)$ on C if and only if $a\theta^2 + b\theta + b\ln(1-\theta) > 0$ and, under this condition, the one-parameter family of quasi-stationary distributions $\{(m_i(\lambda); i \in C); \lambda \in (0, \lambda_C]\}$ is given by (5.9) with $m_1 = \lambda/a$.

Secondly, if $\theta = 1$, then

$$B(s) = a(1-s) + b(1-s)\ln(1-s), \qquad |s| \leq 1$$

and $B'(1) = +\infty$. Therefore,

$$\lambda_C = a + b(1 + \ln(1 - s_*))$$

where s_* is the unique root of $a + b(s + \ln(1-s)) = 0$ in $(0, 1)$

For any $\lambda \in [0, \lambda_C]$, a λ-invariant measure can be given by

$$M_\lambda(s) = \frac{m_1 a}{B(s) + \lambda s}, \qquad |s| < s_\lambda, \tag{5.10}$$

where $M_\lambda(s) = \sum_{i=1}^\infty m_i(\lambda)s^{i-1}$ and s_λ is the smallest positive root of the equation

$$a(1-s) + b(1-s)\ln(1-s) + \lambda s = 0.$$

Finally, by Theorem 4.2, there does not exist any quasi-stationary distribution for $p_{ij}(t)$ on C.

The following example provides an important and interesting case in which the decay parameter λ_C can NOT be obtained by finding the tangent line of $B(s)$.

Example 5.4. Suppose that $a > 0$ and $\beta \in (0, 1]$. Let

$$b_0 = a,\ b_1 = -a - h(\beta), and\ b_k = \frac{\beta^k}{(k-1)k(k+1)},\ (k \geq 2)$$

where $h(\beta) = \sum_{k=2}^{\infty} \frac{\beta^k}{(k-1)k(k+1)}$. For this queueing model, we have

$$B(s) = a - (a + h(\beta))s + \sum_{k=2}^{\infty} \frac{\beta^k s^k}{(k-1)k(k+1)}, \quad s \in [0, 1/\beta]$$

and thus $\rho = 1/\beta < \infty$. It is easy to get that

$$B(s) - B'(s)s = a - \sum_{k=2}^{\infty} \frac{\beta^k s^k}{k(k+1)} \downarrow a - \frac{1}{2},\ \text{as } s \uparrow 1/\beta.$$

Hence if $a \leq 1/2$ then we may still get the decay parameter by finding the tangent line as

$$\lambda_C = -B'(s_*)$$

where s_* is the unique root of $\sum_{k=2}^{\infty} \frac{\beta^k s^k}{k(k+1)} = a$. However, if $a > 1/2$ then we have $B(\rho) - \rho B'(\rho) > 0$ and thus λ_C can NOT be obtained by finding the tangent line of $B(s)$. Notwithstanding this, by Corollary 2.1 we may still obtain that

$$\lambda_C = -B(1/\beta)\beta.$$

Note that this example also shows that we actually do not need to calculate $B(\rho) - \rho B'(\rho)$ even for the last case of $a > \frac{1}{2}$, which may be a hard job.

Finally, for any $\lambda \in [0, \lambda_C]$, a λ-invariant measure $(m_i(\lambda); i \in C)$ for Q (or for $p_{ij}(t)$) on C is given by

$$M_\lambda(s) = \frac{m_1 a}{B(s) + \lambda s}, \qquad |s| < s_\lambda, \tag{5.11}$$

where $M_\lambda(s) = \sum_{i=1}^{\infty} m_i(\lambda)s^{i-1}$ and s_λ is the smallest positive root of $B(s) + \lambda s = 0$. By Theorem 4.2, there exists a quasi-stationary distribution for $p_{ij}(t)$ on C if and only if $\beta \in (0, 1)$ and $B'(1) < 0$. Under this condition, the one-parameter family of quasi-stationary distributions $\{(m_i(\lambda); i \in C); \lambda \in (0, \lambda_C]\}$ is given by (5.11) with $m_1 = \lambda/a$.

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