ON THE CONVERGENCE RATES TO THE EQUILIBRIUM FOR THE BROWNIAN MOTION WITH DIVERGENCE FREE DRIFTS

Shuenn-Jyi SHEU *Institute of Mathematics, Academia Sinica, Taiwan*, E-mail: sheusj@math.sinica.edu.tw

Abstract: We consider the diffusion process on d-dim torus $\mathbf{T}$,

$$dX^{(c)}(t) = cb(X^{(c)}(t))dt + dB(t),$$

where $B(t)$ is the Brownian motion. We assume $b(\cdot)$ is a smooth vector field with period 1 and $b(\cdot)$ is divergence free, which means $div(b) = 0$. The last condition implies that the Lebegues measure on $\mathbf{T}$ is the invariant measure for $X^{(c)}(t)$. We study the rates of convergence of $X^{(c)}(t)$ to the equilibrium for large c. Our main result is the following.

Let $L^{(c)}$ be the generator of $X^{(c)}(t)$,

$$L^{(c)}f = \frac{1}{2}\Delta f + cb \cdot \nabla f.$$

Define

$$\rho^{(c)} = \inf\{-Re(\rho); \rho \neq 0, \rho \text{ is in the spectrum of } L^{(c)}\}.$$

$\rho^{(c)}$ is used to measure the convergence rate of the diffusion process $X^{(c)}(t)$ to the equilibrium. That is, for some $K(c)$,

$$\int_{\mathbf{T}} |p_t^{(c)}(x, y) - 1| dy \leq K(c) \exp(-\rho^{(c)} t),$$

$p_t^{(c)}(x, y)$ is the transition density of $X^{(c)}(t)$. We show that $\rho^{(c)}$ converges to $\rho^{(\infty)}$ as $c \to \infty$. Here

$$\rho^{(\infty)} = \inf\{\frac{1}{2}\int_{\mathbf{T}} |\nabla\phi(x)|^2 dx\},$$

where the infimum is taken over all $\phi = \phi_1 + i\phi_2$ such that

$$\int_{\mathbf{T}} \phi(x)dx = 0, \int_{\mathbf{T}} |\phi(x)|^2 dx = 1$$

and $b\nabla\phi = i\mu\phi$ for some $\mu \in R$. Some examples to calculate $\rho^{(\infty)}$ will be given.