X_t : Continuous semi martingale

$L_t(x)$: Local time of X_t at level x :

$$L_t(x) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1_{[a,a+\varepsilon)}(X_s)\, d(X)_s$$

Tanaka formula: (Tanaka 1963)

$$|X_t - a| = |X_0 - a| + \int_0^t \mathrm{sgn}(X_s)\, dX_s$$
$$+ 2 L_t(a)$$

Meyer's formula (Meyer 1976)

f: convex function,

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)\,dX_s$$
$$+ \int_{\mathbb{R}} L_t(a)\,df'_-(a)$$

Bouleau and Yor 1981: $f(x)$ is an absolutely continuous function with locally bounded derivative, they introduced $\int_{-\infty}^{+\infty} \nabla f(a)\,dL_t(a)$: for a simple function

$$g(x) = \sum g_i \, 1_{(x_i, x_{i+1}]}(x)$$

$$\int_{-\infty}^{+\infty} g(x)\,dL_t(x) = \sum_i g_i \left(L_t(x_{i+1}) - L_t(x_i) \right)$$

Problem: Convergence

Boulean and Yor's basic idea: by Tanaka formula

$$\sum_i g_i \left((X_t - x_i)^+ - (X_t - x_{i+1})^+ \right)$$

$$= \sum_i g_i \left((x - x_i)^+ - (x - x_{i+1})^+ \right)$$

$$+ \sum_i g_i \int_0^t 1_{x_i < X_s \leq x_{i+1}} \, dX_s$$

$$+ 2 \sum_i g_i \left(L_t(x_{i+1}) - L_t(x_{i+1}) \right)$$

Note

$$\sum_i g_i\left((x-x_i)^+ - (x-x_{i+1})^+ \right)$$

$$= \sum_i g_i \int_{-N}^{x} 1_{(x_i, x_{i+1}]}(x)\,dx$$

$$= \int_{-N}^{x} \sum_i g_i\, 1_{(x_i, x_{i+1}]}(x)\,dx$$

$$\Rightarrow \int_{-N}^{x} g(x)\,dx = f(x)$$

The convergence of $\sum g_i\left(L_t(x_{i+1}) - L_t(x_i)\right)$ is through the convergence of all other terms in Ito formula and

$$\int_{-\infty}^{+\infty} \nabla f(x)\, d_x L_t(x)$$

$$= f(X_t) - f(X_0) - \int_0^t \nabla f(X_s) dX_s$$

Follmer and Protter (2000)

Eisenbaum (2000)

Flandoli, Russo and Wolf (2003)

⋮

Eisenbaum's extension

$$f(t, B_t) = f(0, B_0) + \int_0^t f(B_s, ds)$$

$$+ \int_0^t \frac{\partial}{\partial x} f(s, B_s)\, dB_s$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \frac{\partial f}{\partial x}(s, B_s)\, d_{s,x} \Lambda_s(x)$$

f: absolutely continuous in x with certain growth condition on $\frac{\partial f}{\partial x}$

Here

$$\int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial x} f(s, B_s)\, d_{s,x} L_s(x)$$

$$= \int_0^t \frac{\partial}{\partial x} f(s, B_s)\, dB_s - \int_0^t \frac{\partial}{\partial x} f(s, B_s)\, d^* B_s$$

$$\int_0^t \frac{\partial}{\partial x} f(s, B_s)\, d^* B_s = \lim_{n \to \infty} \sum \frac{\partial f}{\partial x}(t_{i+1}, B_{t_{i+1}}) (B_{t_{i+1}} - B_{t_i})$$

(backward stochastic integral) and for simple function $g(s,x) = \sum_{ij} g_{ij} 1_{(s_i, s_{i+1}]}(s) 1_{(x_j, x_{j+1}]}(x)$

$$\int_0^t \int_{-\infty}^{+\infty} g(s,x)\, dL_s(x)$$

$$= \sum_i \sum_j g_{ij} \Big(L_{s_{i+1}}(x_{j+1}) - L_{s_i}(x_{j+1}) - L_{s_{i+1}}(x_j) + L_{s_i}(x_j) \Big).$$

Remark.

Neither Bouleau and Yor's integral nor Eisenbaum's integral are pathwisely defined. Difficult to use.

Two different directions (also recognised by Oberwolfach Mini-Workshop on Local Time space calculus May 2004)

(i) Theortical existence result of the Ito correction term (no explicit form, application seems difficult)

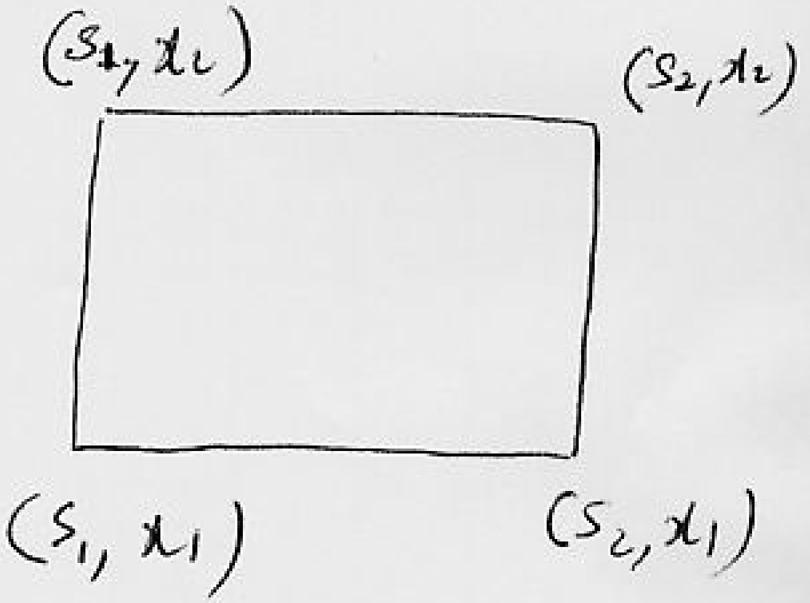
Exaple: Above results

(ii) Explicit formula for the correction term, so application to various problems possible

Tanaka, Meyer, Elworthy + Truman + Z., Feng + Z., Peskir, Azema + Jeulin + Knight + Yor

Lebesgue - Stieltjes integral with local time

$f(s, x)$: monotonely increasing if



Whenever $s_2 \geq s_1$, $x_2 \geq x_1$,

$$f(s_2, x_2) - f(s_2, x_1) - f(s_1, x_2) + f(s_1, x_1) \geq 0.$$

One can define a measure

$$\mu\left((s_1, s_2] \times (x_1, x_2] \right)$$

$$= f(s_2, x_2) - f(s_1, x_2) - f(s_2, x_1)$$

$$+ f(s_1, x_1) .$$

$\mathcal{P}$: a partition of $[t_1, t_2] \times [a, b]$

$$t_1 = s_1 < s_2 < \cdots < s_m = t_2$$

$$a = x_1 < x_2 < \cdots < x_n = b$$

and Variation of f associated with $\mathcal{P}$

$$V_{\mathcal{P}}(f, [t_1, t_2] \times [a, b])$$

$$= \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} | f(s_{i+1}, x_{j+1})$$

$$- f(s_i, x_{j+1}) - f(s_{i+1}, x_j) + f(s_i, x_j)|$$

$$V_f([t_1, t_2] \times [a, b]) = \sup_{\mathcal{P}} V_{\mathcal{P}}(f, [t_1, t_2] \times [a, b])$$

<u>Proposition</u> A function $f(s,x)$ of bounded variation can be decomposed as the difference of two increasing functions $f_1(s,x)$ and $f_2(s,x)$. [c.f. Mcshane]

So for a measurable function g

$$\iint g\,df = \iint g\,df_1 - \iint g\,df_2.$$

<u>Theorem 1</u> (Elworthy, Truman and Zhao)

Assume $\nabla^- f$ is of locally bounded variation in (t,x). Then for every z $(x(0)=z)$

$$f(t, X(t)) = f(0, z) + \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))ds + \int_0^t \nabla^- f(s, X(s))\, d\tilde{s} + \int_{-\infty}^{+\infty} L_t(x)\, d_x \nabla^- f(t, x) - \int_{-\infty}^{+\infty} \int_0^t L_s(x)\, d_{s,x} \nabla^- f(s, x).$$

Krylov's extension of Itô's formula

$f : R^+ \times R \to R$ is C^1 in x and ∇f is absolutely continuous and $\frac{\partial}{\partial s} f(s, x)$, Δf locally in $L^2(ds\,dx)$, then

$$f(0, X(t)) = f(0, z) + \int_0^t \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))\, dX(s) + \frac{1}{2}\int_0^t \Delta f(s, X(s))\, d\langle X\rangle_s$$

Theorem 2 (Elworthy, Truman + Z.)

$f = f_m + f_v$

f_m: C^1 in x (Krylov's condition)

f_v: condition in Theorem 1. Then

$$f(t, X(t)) = f(0, z) + \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))\,ds$$

$$+ \int_0^t \nabla^- f(s, X(s))\,dX(s)$$

$$+ \frac{1}{2}\int_0^t \Delta f_m(s, X(s))\,d\langle X\rangle_s$$

$$+ \int_{-\infty}^{+\infty} L_t(x)\,d\nabla^- f_v(t, x)$$

$$- \int_{-\infty}^{+\infty}\int_0^t L_s(x)\,d_{s,x}\nabla^- f(s, x) \quad \text{a.s.}$$

If $f(t,x)$ has discontinuous derivative along a curve $x = \gamma(t)$ (γ: continuous and of bounded variation). Define

$$X^*(s) = X(s) - \gamma(s)$$

and $L_s^*(x)$: local time of X^*. (Local time on a curve)

<u>Theorem 3</u> (E-7-2). If $\Delta^- f_v$ is well defined every where then

$$
\begin{aligned}
f(t, X(t)) = f(0, z) &+ \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))\, ds \\
&+ \int_0^t \nabla^- f(s, X(s))\, dX(s) \\
&+ \frac{1}{2}\int_0^t \Delta^- f(s, X(s))\, dX(s) \\
&+ \int_0^t \left(\nabla f(s, \gamma(s)+) - \nabla f(s, \gamma(s)-)\right) dL_s^*(0)
\end{aligned}
$$

This theorem can be obtained by applying Theorem 2 and the following decomposition

$$f_m(t,x) = f(t,x) + \left(\nabla f(t,\gamma(t)-) - \nabla f(t,\gamma(t)+)\right)(x-\gamma(t))^+$$

$$f_0(t,x) = \left(\nabla f(t,l(t)+) - \nabla f(t,l(t)-)\right)(x-\cancel{l}\,\gamma(t))^+$$

The Main theorem 2 contains

* Itô formula (classical)
* Tanaka formula
* Meyer formula
* Theorem 3 (obtained indepently by Peskir)
* Azéma, Jeulin, Knight and Yor

Generalization to 2-dimension

$X(t) = (X_1(t), X_2(t))$ 2-dim cts

semimartingale. No local time

Observation: formally

$$\frac{1}{2}\int_0^t \Delta_1 f(s, X_1(s), X_2(s))\, d\langle X_1\rangle_s$$

$$= \int_{-\infty}^{+\infty}\int_0^t \Delta_1 f(s, a, X_2(s))\, d_s L_s^1(a)\, da$$

$$= \int_{-\infty}^{+\infty} \Delta_1 f(t, a, X_2(t))\, L_t^1(a)\, da$$

$$- \int_{-\infty}^{+\infty}\int_0^t L^1(s,a)\, d_{s,a} \underbrace{\nabla_1 f(s, a, X_2(s))}_{\text{semimartingale.}}$$

if the last integral is well defined.

We need to define

$$\int_0^t \int_{-\infty}^{+\infty} g(s,x)\, d_{s,x}\, h(s,x)$$

(Stochastic Lebesgue - Stieltjes integral)

Here $h : [0,t] \times (-\infty, +\infty) \times \Omega \to R$

is $\mathcal{B}([0,t] \times R) \times \mathcal{F}$ measurable, ~~h(s,x)~~

$h(s,x)$: cts M_2 martingale in s for fixed x, and $\langle h(x), h(y) \rangle_s$ is of locally bounded variation in (x,y).

For a simple function

$$g(s,x,\omega) = \sum_{i=1}^{n} \sum_{j=1}^{m} g(t_j, x_i)\, 1_{(t_j, t_{j+1}]}(s)\, 1_{(x_i, x_{i+1}]}(x)$$

define

$$I_1 = \int_0^t \int_{-\infty}^{+\infty} g(s.x)\, d_{s.x}\, h(s.x)$$

$$= \sum_i \sum_j g(t_j, x_i) \Big[h(t_{j+1} \wedge t;\, x_{i+1}) - h(s_j \wedge t, x_{i+1}) - h(s_j \wedge t, x_{i+1}) + h(s_j \wedge t, x_i) \Big]$$

<u>Lemma</u> (**Ito** isometry) (Feng + Z.)

$$E\left(\int_0^t \int_{-\infty}^{+\infty} g(s,x)\, d_{s.x}\, h(s,x)\right)^2$$

$$= E \int_0^t \int_{R^2} g(s,x)\, g(s,y)\, d_{x,y}\, d_s \langle h(x), h(y) \rangle_s$$

Where $d_{x,y}\, d_s \langle h(x), h(y) \rangle_s$ is the product Measure in the space $[0,T] \times R^2$ defined as

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} d_{x,y}\, d_s \langle h(x), h(y)\rangle_s$$

$$= \langle h(x_2), h(y_2)\rangle_{t_1}^{t_2} - \langle h(x_2), h(y_1)\rangle_{t_1}^{t_2}$$

$$- \langle h(x_1), h(y_2)\rangle_{t_1}^{t_2} + \langle h(x_1), h(y_1)\rangle_{t_1}^{t_2}$$

With Ito isometry, one can use standard method to extend the stochastic Lebesgue-Stieltjes integral to measurable and $\mathcal{F}_s$ adapted function $f(s, x, \omega)$.

<u>Theorem 4</u> (Feng + Z.)

If $\frac{\partial^-}{\partial t} f$, $\nabla_1^- f$, $\nabla_2^- f$ exist for all (t, x_1, x_2) and $\nabla_i^- f$ is of locally bounded in x_i

$\nabla_1^- \nabla_2^- f$ is of locally bounded variation in (t, x_1) and (t, x_2). Then

$$
\begin{aligned}
& f(t, X_1(t), X_2(t)) \\
&= f(0, z_1, z_2) + \int_0^t \frac{\partial^-}{\partial s} f(s, X_1(s), X_2(s))\, ds \\
&\quad + \sum_{i=1}^{2} \int_0^t \nabla_i^- f(s, X_1(s), X_2(s))\, dX_i^-(s) \\
&\quad + \int_{-\infty}^{+\infty} L^1(t,a)\, d_a \nabla_1^- f(t, a, X_2(t)) \\
&\quad - \int_{-\infty}^{+\infty} \int_0^t L^1(s,a)\, d_{s,a} \nabla_1^- f(s, a, X_2(s)) \\
&\quad + \int_{-\infty}^{+\infty} L^2(t,a)\, d_a \nabla_2^- f(t, X_1(t), a) \\
&\quad - \int_{-\infty}^{+\infty} \int_0^t L^2(s,a)\, d_{s,a} \nabla_2^- f(s, X_1(s), a) \\
&\quad + \int_0^t \nabla_1^- \nabla_2^- f(s, X_1(s), X_2(s))\, d\langle M_1, M_2\rangle_s
\end{aligned}
$$

Rough path integration of local time

Definition A continuous path $X:[0,T] \to R$ is said to have finite p-variation if

$$\sup_{\mathcal{D}} \sum_{l} |X_{t_l} - X_{t_{l-1}}|^p < \infty$$

where $\sup_{\mathcal{D}}$ runs over all finite division of $[0,T]$.

If X : finite p-Variation

Y : finite q-Variation

$\frac{1}{p} + \frac{1}{q} = \theta > 1$

$$\sum_{\ell} Y_{t_{\ell-1}} (X_{t_\ell} - X_{t_{\ell-1}})$$

$$\to \int_0^T Y_t \, dX_t$$

(Lyons Rough Path Integration)

Our observations

<u>Known facts</u>

* Local time $L_t(x)$ is continuous and increasing in t

* $L_t(x)$ is of quadratic variation in a, has compact support

* $L_t(x)$ is RCLL in x, but ~~all~~ total variation of

the jumps $L_t(a)-L_t(a-)$ is of bounded variation. Denote $\{a_k^*\}$ ~~the~~ all the jumps of $L_t(\cdot)$.

Decomposition of Local time

$$L_t(a) = \tilde{L}_t(a) + \sum_k \hat{L}_t(a_k^*) 1_{a \geq a_k^*}$$

. Here $\tilde{L}_t(a)$ continuous in a.

One can prove for any partition $\mathcal{D}$ of $[-N, N]$ ($[-N, N]$ contains the supports of $L_t(a)$)

$$\lim_{m(\mathcal{D})\to 0} \sum_i \left(\tilde{L}_t(a_{i+1}) - \tilde{L}_t(a_i)\right)^2$$

$$= 2 \int_{-N}^{N} L_t(x)\,dx$$

Therefore one can define

$$\int_{-\infty}^{+\infty} f(a)\, d_a L_t(a)$$

$$= \int_{-\infty}^{+\infty} f(a)\, d_a \tilde{L}_t(a) \quad \text{(rough path integration)}$$

$$+ \int_{-\infty}^{+\infty} f(a)\, d_a \hat{L}_t(a) \quad \text{(L-S integral)}$$

Theorem 5 (Feng + Z.)

$f(x)$: locally finite q-variation and continuous

$$f_n(x) = \int_0^2 \rho(z)\, f(x - \tfrac{z}{n})\,dz$$

$g(x)$: p-variation

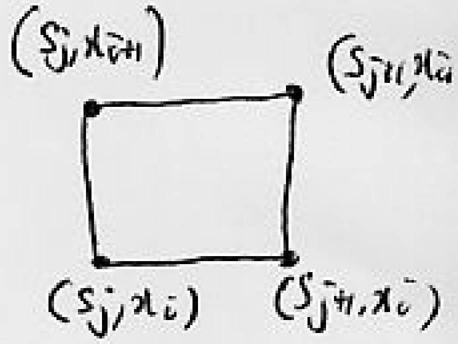
then as $n \to \infty$

$$\int_{-\infty}^{+\infty} f_n(x)\, dg(x) \to \int_{-\infty}^{+\infty} f(x)\, dg(x)$$

Integration with respect to (s, x). For a partition of $[0, t] \times [-N, N]$

$$\lim_{m(\Delta)\to 0} \sum_{i,j} \left| \tilde{L}_{s_j}^{s_{j+1}}(x_{i+1}) - \tilde{L}_{s_j}^{s_{j+1}}(x_i) \right|^2$$

$$= 2 \int_{-N}^{N} L_t(a)\, da$$



One can use the point deleting idea of Lyons to define

$$\int_0^t \int_{-\infty}^{+\infty} g(s, x)\, d_{s,x} \tilde{L}_s(x)$$

Define

$$f_n(s,x) = \int_0^2 \int_0^2 \rho(t)\rho(z) f(s-\tfrac{t}{n}, x-\tfrac{z}{n}) \, dt\,dz$$

Theorem 6 (Feng + Z.)

As $n \to \infty$,

$$\int_{-\infty}^{+\infty} \int_0^t f_n(s,x)\, d_{s,x} L_s^x$$

$$\to \int_{-\infty}^{+\infty} \int_0^t f(s,x)\, d_{s,x} L_s^x \quad \text{a.s.}$$

Application in the asymptotics of heat equation

Consider

$$\begin{cases} \frac{\partial}{\partial t} u^{\varepsilon}(t,x) = \frac{\varepsilon^2}{2}\Delta u^{\varepsilon} + \frac{1}{\varepsilon^2} c(x) u^{\varepsilon} \\ u^{\varepsilon}(0,x) = T_0(x)\, e^{-\frac{S_0(x)}{\varepsilon^2}} \end{cases}$$

Large deviation: (Varadhan, Donsker, Wentzell Freidlin, et. al)

$$\varepsilon^2 \log u^{\varepsilon}(t,x) \longrightarrow -S(t,x) \quad \text{as } \varepsilon \to 0$$

Here $S(t,x)$ is the (Weak) solution of the Hamilton Jacobi equation

$$\begin{cases} \frac{\partial}{\partial t} S + \frac{1}{2}|\nabla S|^2 + c(x) = 0 \\ S(0,x) = S_0(x) \end{cases}$$

Under a "no-caustics" condition, (i.e. $S(t,x)$ is a $C^{1,2}$ solution of the H-J equation)

let X_s^ε be a solution of the stochastic differential equation

$$dX_s^\varepsilon = \varepsilon \, dB_s - \nabla S(t-s, X_s^\varepsilon)\, ds$$

Elworthy and Truman (82)

$$u^\varepsilon(t,x) = e^{-\frac{S(t,x)}{\varepsilon^2}} \, \mathbb{E}\, T_0(X_t^\varepsilon)\, e^{-\frac{1}{2}\int_0^t \Delta S(t-s, X_s^\varepsilon)\, ds}$$

$$u^\varepsilon(t,x)\, e^{\frac{S(t,x)}{\varepsilon^2}} \to T_0(\Phi_t^{-1}x)\, e^{-\frac{1}{2}\int_0^t \Delta S(\Phi_s \Phi_t^{-1} x)\, ds}$$

There are many applications.

<u>Open problems was</u>: Semi-classical representation and asymptotics in the presence of caustics.

The generalized Itô's formula that we have developed has helped us to solve the long standing problem in one dimension (Elworthy, Truman + Z.)

Thank you!!