Perturbation of symmetric Markov processes

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This talk is based on the joint works with T. J. Lyons, J. Lunt, Z. Q. Chen, P. J. Fitzsimmons and K. Kuwae.

§1. Forward and backward Girsanov transformations

Let $a = (a_{ij}(x))_{1 \le i,j \le d}$ be a $d \times d$ positive definite symmetric matrices-valued measurable function on R^d such that

$$\delta|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i \xi_j \leq \frac{1}{\delta}|\xi|^2, \quad \xi \in R^d$$

where δ is a positive constant.

Let $(\Omega, \mathcal{F}, X_t, \theta_t, P_x, x \in R^d)$ denote the diffusion process generated by

$$\mathcal{L}_0 f = \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial f}{\partial x_i}\right).$$

The Dirichlet form associated with the diffusion process is given by

$$\mathcal{E}_0(u,v) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx$$

$$\mathcal{D}(\mathcal{E}_0) = H_2^1(R^d)$$
$$= \left\{ u \in L^2(R^d) \,;\, \frac{\partial u}{\partial x_i} \in L^2(R^d), \right.$$
$i=1,\dots,d \}$

Given a vector field

$$b = (b_1(x), b_2(x), \cdots, b_d(x))$$

on R^d. Consider the drift perturbation

$$\mathcal{L}_1 = \mathcal{L}_0 + \langle b, \nabla \cdot \rangle$$

Then the diffusion process associated with $\mathcal{L}_1$ can be obtained from the diffusion process associated with $\mathcal{L}_0$ through a Girsanov transform.

To illustrate the idea, without loss of generality, we assume from now on $a = (a_{ij}(x)) = I_{d \times d}$ is the identity matrix.

Let $(\Omega, \mathcal{F}, X_t, \theta_t, Q_x, x \in R^d)$ be the diffusion associated with

$$\mathcal{L}_1 = \frac{1}{2}\Delta + \langle b, \nabla \cdot \rangle .$$

Then it is well known that

$$\frac{dQ_x}{dP_x}\bigg|_{\mathcal{F}_t} = \exp\left\{ \int_0^t b(X_s)\, dB_s - \frac{1}{2}\int_0^t |b(X_s)|^2 ds \right\},$$

where $(B_t, t \geq 0)$ is a Brownian motion.

Now given a second vector field

$$\hat{b}(x) = (\hat{b}_1(x), \hat{b}_2(x), \dots, \hat{b}_d(x))$$

and a measurable function $c(x)$ on R^d. Consider the following perturbation of the Laplacian operator :

$$Lf = \frac{1}{2}\Delta f + \langle b, \nabla f \rangle - \operatorname{div}(\hat{b} f) + c(x) f(x).$$

The associated quadratic form is given by

$$Q(u,v) = \frac{1}{2}\int_{R^d} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx - \int_{R^d} \langle b, \nabla u\rangle(x)\, v(x)\,dx - \int_{R^d} \langle \hat{b}(x), \nabla v(x)\rangle\, u(x)\,dx - \int_{R^d} c(x)\,u(x)\,v(x)\,dx$$

It is not immediately clear how to give a ~~A~~ probabilistic representation of the semigroup T_t associated with the generator $(L, D(L))$.

Here is the result :

<u>Theorem 1</u> (Lyons, TZ) ;

The semigroup T_t is given by

$$
\begin{aligned}
&T_t g(x) \\
&= E_x\Big[g(B_t) \exp\Big(\int_0^t \langle b(B_s), dB_s\rangle \\
&\quad + \int_0^t \langle \hat{b}(B_{t-s}), dB_s \circ \gamma_t \rangle \\
&\quad - \tfrac{1}{2} \int_0^t |b-\hat{b}|^2(B_s)\,ds + \int_0^t c(B_s)\,ds \Big)\Big],
\end{aligned}
$$

where $\gamma_t(\cdot)$ is the reverse operator defined by

$$B_s(\gamma_t(w)) = B_{t-s}, \quad s \le t.$$

<u>Remark</u>. We can see immediately that T_t is positivity preserving.

§2: Girsanov and Feynman-Kac transformations.

Now we move to the general setting of symmetric Markov processes.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E, m)$, where E is a locally compact separable metric space. and m is a σ-finite Borel measure.

Denote by

$$X = (\Omega, \mathcal{F}, X_t, \theta_t, P_x, x \in E)$$

the Hunt process associated with $(\mathcal{E}, \mathcal{F})$.

It is well known that the following Beurling-Deny decomposition holds

$$\mathcal{E}(f,g) = \mathcal{E}^{(c)}(f,g)$$
$$+ \int_{E\times E\setminus d} (f(x)-f(y))(g(x)-g(y))\, J(dx,dy)$$
$$+ \int_E f(x)g(x)\, k(dx),$$

where $\mathcal{E}^{(c)}$ is the strongly local part.

For $u \in \mathcal{F}$, the following Fukushima's decomposition holds.

$$u(X_t) - u(X_0) = M_t^u + N_t^u,$$

where N_t^u is a continuous additive functional of X having zero quadratic variation, M^u is a martingale additive functional of X. which can be further decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,k}$$

corresponding to the jumping part, continuous part and the killing part of M^u.

Denote by $\mu_{\langle u \rangle}$, $\mu^c_{\langle u \rangle}$, $\mu^j_{\langle u \rangle}$ and $\mu^k_{\langle u \rangle}$ the Revuz measures associated with the sharp bracket. $\langle M^u \rangle$, $\langle M^{uc} \rangle$, $\langle M^{uj} \rangle$ and $\langle M^{uk} \rangle$.

We are interested in the following Feynman-Kac type transformation of X:

$$\hat{P}_t f(x) = E_x[f(X_t)\, e^{N_t^u}]$$

and its characterization.

When $u \in D(L)$, then

$$N_t^u = \int_0^t Lu(X_s)\,ds .$$

~~The~~ $\hat{P}_t$ is a Feynman-Kac transform in this case. But in general N^u is not of finite variations. So the classical result for Feynman-Kac transform do not apply.

For example, the Hilbert transform of Brownian local times is N^u for $u(x) = x\log|x| - x$.

Here is

[Z.Q. Chen, TZ]

<u>Theorem 2</u>. Assume that the function u in $\mathcal{F}$ such that $\mu_{\langle u \rangle}$ is in Kato class of X. Then $\hat{P}_t$, $t \geq 0$

is a strongly continuous symmetric semigroup on $L^2(E, m)$. Let $(Q, D(Q))$ be the quadratic form associated with $\hat{P}_t$ on $L^2(E, m)$. Then $D(Q) = \mathcal{F}$ and for $f, g \in \mathcal{F}_b$,

$$Q(f, g) = \mathcal{E}(f, g) + \mathcal{E}(u, fg).$$

When X is Brownian motion on R^n, Theorem 2 was established by Glover, Song et al under an additional assumptio[n] that u is bounded using an approximati[on] method that fails to work in the general situation. To explain our approach, assume for the moment X is continuou[s]. Rewrite $\hat{P}_t$ as

$$\hat{P}_t f(x) = E_x[f(X_t) e^{N_t^u}]$$

$$= E_x[f(X_t) e^{u(X_t) - u(X_0) - M_t^u}]$$

$$= E_x[(f e^u)(X_t) e^{-M_t^u}] e^{-u(x)}$$

$$= e^{-u(x)} E_x[(f e^u)(X_t) e^{-M_t^u - \frac{1}{2}\langle M^u \rangle_t} \cdot e^{\frac{1}{2}\langle M^u \rangle_t}]$$

$$= e^{-u(x)} E_x[(f e^u)(X_t) L_t e^{\frac{1}{2}\langle M^u \rangle_t}],$$

where $L_t = \exp(-M_t^u - \frac{1}{2}\langle M^u \rangle_t)$.

So the transform is a result of a Girsanov transform by the exponential martingale L_t followed by a Feynman-Kac transform $\exp(\frac{1}{2}\langle M^u \rangle_t)$. So we need to study the Girsanov transform of symmetric Markov processes. To this end, fix a bound function $u \in \mathcal{F}$ and let $\rho(x) = e^{u(x)}$. By Fukushima's decomposition

$$\rho(X_t) - \rho(X_0) = M_t^\rho + N_t^\rho,$$

where M^ρ is a martingale additive functional. Define

$$M_t = \int_0^t \frac{1}{\rho(X_{s-})} \, dM_s^\rho$$

($M_t = M_t^u$ if X is continuous)

Let L_t^ρ denote the solution of the SDE

$$L_t^\rho = 1 + \int_0^t L_{s-}^\rho \, dM_s.$$

L^ρ is given by

$$L^\rho_t$$

$$= \exp\left(M_t - \tfrac{1}{2}\langle M^{uc}\rangle_t\right) \times \prod_{0<s\le t} \frac{\rho(X_s)}{\rho(X_{s-})} \exp\left(1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right).$$

Define a family of new probability measures by

$$\left.\frac{d\widetilde{P}_x}{dP_x}\right|_{\mathcal{F}_t} = L^\rho_t.$$

Then under the new measures $\widetilde{P}_x$, X is a $\rho^2 dm$-symmetric, right Markov process. Next Theorem identifjes the Dirichlet form associated with the new process denoted by $\widetilde{X}$.

[Z.Q. Chen, T2]

Theorem 3. Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be the Dirichlet form of $\tilde{X}$ on $L^2(E, \rho^2 dm)$. Then $\tilde{\mathcal{F}} = \mathcal{F}$ and for $f \in \tilde{\mathcal{F}}$,

$$\tilde{\mathcal{E}}(f,f) = \frac{1}{2}\int_E \rho(x)^2 \mu^c_{\langle f \rangle}(dx)$$

$$+ \int_{E \times E \setminus d} (f(x) - f(y))^2 \rho(x)\rho(y) J(dx, dy)$$

$$+ \int_E f(x)^2 \rho(x) \kappa(dx)$$

§3: Perturbations of symmetric Markov processes.

In this section I will present the main result of a recent joint work with Z. Q. Chen, P. Fitzsimmons and Kuwae, where we extended the results of Lyons and TZ to general martingale additive functionals and general symmetric Markov processes.

We use the same framework as in Section 2. Recall that $\mathcal{X} = (\Omega, \mathcal{F}, X_t, \theta_t, P_x, x \in E)$ is the Hunt process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Let M, $\hat{M}$ be two locally square integrable martingale additive functionals of X. Let $\mu_{\langle M \rangle}$, $\mu_{\langle \hat{M} \rangle}$ be the smooth measures associated with $\langle M \rangle_t$ and $\langle \hat{M} \rangle_t$.

Let μ be a signed smooth measure associated with a continuous additive functional A^μ. It is known that there are Borel measurable functions φ and ψ on $E \times E_\partial$ vanishing on the diagonal such that

$$M_t - M_{t-} = \varphi(X_{t-}, X_t)$$

$$\hat{M}_t - \hat{M}_{t-} = \psi(X_{t-}, X_t)$$

Write Exp for the Doléans-Dade exponential.

Define a multiplicative function Z by

$$Z_t$$

$$= \mathrm{Exp}(\hat{M}_t)\circ\gamma_t \, \mathrm{Exp}(M_t + A_t^\mu + \langle M^c, \hat{M}^c \rangle_t)$$

$$\cdot (1 + \psi(X_t, X_{t-}))$$

and

$$T_t f(x) = E_x[\, Z_t f(X_t) \,]$$

Our main result reads as

[ZC, PF, KK, TZ]

<u>Theorem 4</u>. Assume that $\mu_{\langle M \rangle}$, $\mu_{\langle \hat{M} \rangle}$ and $|\mu|$ are all in the Kato class $K(X)$. Then $\{T_t, t \geq 0\}$ is a strongly continuous semigroup on $L^2(E, m)$ associated with the following quadratic form $(Q, D(Q))$.

$$D(Q) = \mathcal{F}$$

$$Q(f,g) = \mathcal{E}(f,g) - \int_E g \, d\mu_{\langle M^f, M\rangle}$$

$$- \int_E f \, d\mu_{\langle M^g, \hat{M}\rangle} - \int_E fg \, d\mu$$

$$- \int_{E\times E} f(x) g(x) \varphi(x,y) \psi(y,x) N(x,dy) \mu_H(dx),$$

where μ_H is the smooth measure associated with the PCAF H_t, appeared in the Lévy system of X : $(N(x,dy), H_t)$.

Recall that

$$E_x\left(\sum_{s\le t} \phi(X_{s-}, X_s) \right)$$

$$= E_x\left(\int_0^t \int_{E_\partial} \phi(X_s, y) N(X_s, dy) \, dH_s \right)$$

Key to the proof is to establish the following formula:

For $f \in D(L^{Q})$,

$$Z_t f(X_t) = f(x_0) + \int_0^t Z_{s-}(dM_s^f + dU_s^f)$$
$$+ \int_0^t Z_{s-} f(X_{s-})\, d(M_s^c - \hat{M}_s^c + L_s)$$
$$+ \int_0^t Z_s L^{Q} f(X_s)\, ds,$$

where L and U^f are purely discontinua local MAFs of X with

$$L_t - L_{t-}$$
$$= \varphi(X_{t-}, X_t) + \psi(X_t, X_{t-})$$
$$+ \varphi(X_{t-}, X_t)\, \psi(X_t, X_{t-})$$

and

$$U_t^f - U_{t-}^f = (f(X_t) - f(X_{t-}))(L_t - L_{t-})$$

Since Z_t and $f(X_t)$ are not semi-martingales, ordinary Itô's formula can not be applied. to establish above formula. We need to develop a stochastic calculus for Dirichlet processes, which is of independent interests.

Particularly, we introduced stochastic integral against Dirichlet processes and established the following Itô's formula

<u>Theorem 5</u>: Suppose that $\Phi \in C^2(R^d)$ and $u = (u_1, u_2, \cdots, u_d) \in \mathcal{F}^d$. Then

$$\Phi(u_1(X_t), \cdots u_d(X_t)) - \Phi(u(X_0))$$

$$= \sum_{k=1}^{d} \int_0^t \frac{\partial \Phi}{\partial x_k}(u(X_{s-})) \circ du_k(X_s)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_{s-}))\, d\langle M^{u_i,c}, M^{u_j,c}\rangle_s$$

$$+ \sum_{0<s\le t} \Big(\Phi(u(X_s)) - \Phi(u(X_{s-}))$$

$$- \sum_{k=1}^{d} \frac{\partial \Phi}{\partial x_k}(u(X_{s-}))(u_k(X_s) - u_k(X_{s-}))\Big)$$

Our results extend Nakao's results on stochastic integrals against Dirichlet processes significantly.