Continuous-Time Mean–Risk Portfolio Selection

Hanqing Jin, Xun Yu Zhou
The Chinese University of Hong Kong, Shatin, Hong Kong
Jia-An Yan
Chinese Academy of Sciences, Beijing, P. R. China.

0.1 Introduction

In 1952, Markowitz proposed the Mean-Variance portfolio selection problem for a single period. Recently, the Mean-Variance framework were studied in multi-period market by Li and Ng [4] and in continuous-time market by Zhou and Li [8] and Bielecki, Plisca, Jin, and Zhou [1], among others. In this paper, we will discuss the Mean-Risk problem in a complete continuous time financial market.

Outline

- Problem Formulation
- The Weighted Mean–Variance Model
- The Mean–Semivariance Model
- The Mean–Downside-risk Model
- The General Mean–Risk Model
- Concluding Remarks

0.2 Problem Formulation

Bank account:

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T], \qquad S_0(0) = s_0 > 0.$$

m assets stocks:

$$\begin{cases} dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)], \quad t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases}$$

where $b_i(t)$ and $\sigma_{ij}(t)$ are assumed to be $\mathcal{F}_t$-adapted and uniformly bounded, and $\sigma(t)\sigma(t)' \geq \delta I_m, \quad \forall t \in [0,T], \ \text{a.s.}$, for some $\delta > 0$. This assumption ensures that the market is complete.

Set $B(t) := (b_1(t) - r(t), \cdots, b_m(t) - r(t))'$, and define the *risk premium process* $\theta(t) \equiv (\theta_1(t), \cdots, \theta_m(t)) := B(t)(\sigma(t)')^{-1}$, and the pricing kernel

$$\rho(t) := \exp\left\{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)dW(s)\right\}. \tag{1}$$

With this notation, The wealth $x(t)$ at time $t \geq 0$ of a self-financing strategy $\pi(\cdot) \equiv (\pi_1(\cdot), \cdots, \pi_m(\cdot))'$ satisfies

$$dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t). \quad (2)$$

where $\pi_i(t), \quad i = 0, 1, 2 \cdots, m,$ denotes the total market value of the agent's wealth in the i-th asset at time t.

Definition 0.2.1 *A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}}([0, T]; \mathbb{R}^m)$.*

We denote by $\mathcal{A}(x)$ the set of admissible portfolios with initial wealth x. The wealth process of an admissible portfolio satisfies

$$x(t) = \rho(t)^{-1} E(\rho(T)x(T)|\mathcal{F}_t), \quad \forall t \in [0, T], \text{ a.s.}. \qquad (3)$$

Consider a general portfolio selection problem:

$$\begin{aligned} &\text{Minimize} \quad Ef(x(T)), \\ &\text{subject to } \pi \in \mathcal{A}(x_0),\ x(T) \in D, \end{aligned} \qquad (4)$$

where $D \subset L^2(\mathcal{F}_T, \mathbb{R})$ represents some additional constraints on the terminal wealth $x(T)$, and $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given. Notice that here $-f$ is not necessary a utility function.

The portfolio selection problem (4) can be decomposed into a static optimization problem and a replication problem. The static optimization problem is

$$\begin{aligned} &\text{Minimize} \quad Ef(X), \\ &\text{subject to} \;\; E[\rho(T)X] = x_0, \;\; X \in D. \end{aligned} \tag{5}$$

Suppose X^* is an optimal solution to (5), then the replication problem is to find $(x(\cdot), \pi(\cdot))$ that solves the following BSDE:

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\ x(T) = X^*. \end{cases} \tag{6}$$

Theorem 0.2.1 *If $(x^*(\cdot), \pi^*(\cdot))$ is optimal for problem (4), then $x^*(T)$ is optimal for problem (5) and $(x^*(\cdot), \pi^*(\cdot))$ satisfies (6). Conversely, if X^* is optimal for problem (5), then the solution of (6) $(x^*(\cdot), \pi^*(\cdot))$ is an optimal solution for (4).*

0.3 The Weighted Mean–Variance Model

Ror given $\alpha > 0, \beta > 0, z \in \mathbb{R}, x_0 \in \mathbb{R}$, we consider the "weighted" mean–variance portfolio selection problem:

$$\begin{aligned} &\text{Minimize} \quad E[\alpha(x(T) - z)_+^2 + \beta(x(T) - z)_-^2], \\ &\text{subject to } \pi \in \mathcal{A}(x_0), \;\; Ex(T) = z. \end{aligned} \tag{7}$$

Define $Y := X - z$, then the problem (5) is reduced to the problem in terms of Y:

$$\begin{array}{ll} \text{Minimize} & E(\alpha Y_+^2 + \beta Y_-^2), \\ \text{subject to} & \begin{cases} EY = 0, \\ E[\rho Y] = y_0, \\ Y \in L^2(\mathcal{F}_T, \mathbb{R}), \end{cases} \end{array} \tag{8}$$

where $\rho := \rho(T)$ and $y_0 := x_0 - zE\rho$.

Theorem 0.3.1 *The unique optimal solution for problem*

(8) is

$$Y^* = \frac{(\lambda - \mu\rho)_+}{\alpha} - \frac{(\lambda - \mu\rho)_-}{\beta}$$

where (λ, μ) *is the unique solution of the system of equations:*

$$\begin{cases} \frac{E(\lambda-\mu\rho)_+}{\alpha} - \frac{E(\lambda-\mu\rho)_-}{\beta} = 0 \\ \frac{E[\rho(\lambda-\mu\rho)_+]}{\alpha} - \frac{E[\rho(\lambda-\mu\rho)_-]}{\beta} = y_0. \end{cases} \tag{9}$$

The minimum value of the problem (8) is

$$E[\alpha(Y_+^*)^2 + \beta(Y_-^*)^2] = -\mu y_0. \tag{10}$$

0.4 The Mean–Semivariance Model

Now we consider the mean–semivariance problem, which is to

$$\begin{aligned} &\text{Minimize} \quad E[(x(T)-z)_-^2], \\ &\text{subject to } \pi \in \mathcal{A}(x_0),\ Ex(T)=z. \end{aligned} \tag{11}$$

Theorem 0.4.1 *The mean–semivariance problem (11) does not admit an optimal solution so long as* $z \neq \frac{x_0}{E\rho}$.

Proof: In view of Theorem 0.2.1, it suffices to prove that the static optimization problem

$$\text{Minimize} \quad E(Y_-^2),$$
$$\text{subject to} \quad \begin{cases} EY = 0, \\ E[\rho Y] = y_0 \equiv x_0 - zE\rho, \\ Y \in L^2(\mathcal{F}_T, \mathbb{R}), \end{cases} \tag{12}$$

has no optimal solution. Consider problem (8) with $\beta = 1 - \alpha$ and $\alpha \in (0,1)$. We denote by $Y(\alpha)$ the corresponding solution. We can prove that as $\alpha \downarrow 0$,

$$E[Y(\alpha)_-^2] = \frac{\mu(\alpha)^2 E(\zeta(\alpha) - \rho)_-^2}{(1-\alpha)^2} \to y_0^2 / E(\rho - \rho_0)^2. \quad (13)$$

On the other hand, for any feasible solution Y of problem (12), Cauchy–Schwartz's inequality yields $\{E[(\rho-\rho_0)Y_-]\}^2 \leq E[Y_-]^2 E[(\rho - \rho_0)^2 \mathbf{1}_{Y<0}]$. Note that $E[(\rho - \rho_0)^2 \mathbf{1}_{Y<0}] \neq 0$, for otherwise $P(Y \geq 0) = 1$ which together with $EY = 0$

implies $P(Y = 0) = 1$ and hence $y_0 = 0$. As a result,

$$\begin{cases} E[Y_-]^2 \geq \frac{\{E[(\rho-\rho_0)Y_-]\}^2}{E[(\rho-\rho_0)^2 \mathbf{1}_{Y<0}]} \\ = \frac{\{E[(\rho-\rho_0)Y_+]-y_0\}^2}{E[(\rho-\rho_0)^2 \mathbf{1}_{Y<0}]} > \frac{y_0^2}{E(\rho-\rho_0)^2}, \end{cases} \tag{14}$$

where the last *strict* inequality is due to the facts that $y_0 < 0$ and $EY = 0$.

Remark 0.4.1 The infimum of the problem is finite and asymptotically optimal portfolios can be obtained by replicating $Y(\alpha)$ as $\alpha \to 0$.

0.5 The Mean–Downside-risk Model

In this section, we generalize the "negative" result obtained in Section 4 to a model with a general downside risk. The risk is measured by a non-negative function f on $\mathbb{R}$, which is strictly decreasing on $\mathbb{R}^-$, and $f(x) = 0 \; \forall x \in \mathbb{R}^+$. An example is $f(x) = (x_-)^p$ for some $p \geq 0$.

Assumption 0.5.1 For any $0 \leq a < b \leq +\infty$, $P\{\rho(T) \in (a, b)\} > 0$ and $P\{\rho(T) = a\} = 0$.

Now we turn to the following mean–downside-risk portfolio selection problem. For each $z \in \mathbb{R}$:

$$\begin{array}{ll} \text{Minimize} & Ef(x(T) - Ex(T)), \\ \text{subject to} & \pi \in \mathcal{A}(x_0), \; Ex(T) = z. \end{array} \tag{15}$$

The corresponding static optimization problem of (15) is

$$\text{Minimize} \quad Ef(X-z),$$
$$\text{subject to} \quad \begin{cases} EX = z, \\ E[\rho(T)X] = x_0, \\ X \in L^2(\mathcal{F}_T, \mathbb{R}). \end{cases} \tag{16}$$

Theorem 0.5.1 *Problem (15) admits no optimal solution for any* $z \neq \frac{x_0}{E\rho(T)}$. *On the other hand, when* $z = x_0/E\rho(T)$, *(15) has an optimal portfolio which is the risk-free portfolio.*

0.6 The General Mean–Risk Model

An interesting problem is the following: for a general convex function f which measures the risk, when the problem (15) does possess an optimal solution?

Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ be a strictly positive real random variable on it satisfying

$$P\{\xi \in (a_1, a_2)\} > 0, P\{\xi = a_1\} = 0, \forall 0 \leq a_1 < a_2 \leq \infty. \tag{17}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex (hence continuous) function, not necessarily differentiable. For any $x \in \mathbb{R}$, its subdifferential $\partial f(x)$ in the sense of convex analysis, is defined as the set $\partial f(x) := [f'_-(x), f'_+(x)]$.

For a given $q \geq 1$ and $y_0 \in \mathbb{R}$, consider the following optimization problem:

$$\text{Minimize} \quad Ef(Y),$$
$$\text{subject to} \quad \begin{cases} EY = 0, \\ E[\xi Y] = y_0, \\ Y \in L^q(\mathcal{F}, \mathbb{R}). \end{cases} \tag{18}$$

According to [1, Proposition 4.1], (18) admits an optimal solution Y^* if and only if Y^* is feasible for (18) and there exists a pair (λ, μ) such that Y^* solves the following problem

$$\min_{Y \in L^q(\mathcal{F}_T, \mathbb{R})} E[f(Y) - (\lambda - \mu\xi)Y]. \tag{19}$$

Lemma 0.6.1 *$Y^* \in L^q(\mathcal{F}_T, \mathbb{R})$ is an optimal solution to (19) if and only if*

$$f(Y^*) - (\lambda - \mu\xi)Y^* = \min_{y \in \mathbb{R}}[f(y) - (\lambda - \mu\xi)y], \quad \text{a.s..}$$

Define a set-valued function G: $\cup_{x \in \mathbb{R}} \partial f(x) \to 2^{\mathbb{R}}$

$$G(y) := \{x \in \mathbb{R} : y \in \partial f(x)\}, \quad \forall y \in \cup_{x \in \mathbb{R}} \partial f(x),$$

and define g: $\cup_{x\in\mathbb{R}}\partial f(x) \to \mathbb{R}$ as the "inverse function" of ∂f as follows

$$g(y) := argmin_{x\in G(y)}|x|, \quad \forall y \in \cup_{x\in\mathbb{R}}\partial f(x).$$

g is a well-defined function (on its domain), and the set of y's where $G(y)$ is not a singleton is countable.

We will solve the problem in each of the following four (mutually exclusive) cases:

Case 1: The set $\cup_{x\in\mathbb{R}}\partial f(x)$ is upper bounded but not lower bounded;

Case 2: The set $\cup_{x\in\mathbb{R}}\partial f(x)$ is lower bounded but not upper bounded;

Case 3: $\cup_{x\in\mathbb{R}}\partial f(x) = \mathbb{R}$;

Case 4: The set $\cup_{x\in\mathbb{R}}\partial f(x)$ is both upper and lower bounded.

Let us first focus on Case 1. In this case, $\cup_{x\in\mathbb{R}}\partial f(x)$ is either a closed interval $(-\infty, \bar{k}]$ or an open one $(-\infty, \bar{k})$ where

$$\bar{k} := \lim_{x\to+\infty} f'_+(x) \in \mathbb{R}. \tag{20}$$

Define

$$\begin{cases} \bar{\Lambda} := \{\lambda \in [f'_-(0), \bar{k}] : \exists \mu = \mu(\lambda) \text{ s.t.} g(\lambda - \mu(\lambda)\xi) \in L^q(\mathcal{F}, \mathbb{I} \\ \qquad\qquad Eg(\lambda - \mu(\lambda)\xi) = 0, \;\; \xi g(\lambda - \mu(\lambda)\xi) \in L^1(\mathcal{F}, \mathbb{R})\}, \\ \bar{\lambda} := \sup_{\lambda \in \bar{\Lambda}} \lambda, \\ \tilde{g}(\lambda) := E[\xi g(\lambda - \mu(\lambda)\xi)], \;\; \lambda \in [f'_-(0), \bar{\lambda}). \end{cases} \tag{21}$$

Notice that $\bar{\Lambda} \neq \emptyset$, since $\partial f(0) \subseteq \bar{\Lambda}$. As a result $f'_+(0) \leq \bar{\lambda} \leq \bar{k}$. Also, $[f'_-(0), \bar{\lambda}) \subseteq \bar{\Lambda}$.

Theorem 0.6.1 *Consider Case 1.*

(i) *If* $\bar{\lambda} \notin \bar{\Lambda}$, *then (18) admits an optimal solution if and only if* $y_0 \in (\underline{y}, 0]$, *where* $\underline{y} = \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda)$. *If* $\bar{\lambda} \in \bar{\Lambda}$, *then (18) admits an optimal solution if and only if* $y_0 \in \{\tilde{g}(\bar{\lambda})\} \cup (\underline{y}, 0]$. *If in addition* $\bar{\lambda} < \bar{k}$, *then* $\tilde{g}(\bar{\lambda}) = \underline{y}$.

(ii) *When* $y_0 = 0$, $Y^* := 0$ *is the unique optimal solution to (18).*

(iii) *When* $y_0 < 0$ *and the existence of optimal solution* *is assured,* $Y^* := g(\lambda - \mu(\lambda)\xi)$ *is the unique optimal solution to (18), where* λ *is the unique solution to* $\tilde{g}(\lambda) = y_0$.

As for Case 2, it can be turned into Case 1 by considering $\tilde{f}(x) = f(-x)$. For Case 3, it can be dealt with similarly combining the analyses for the previous two cases.

The final case, Case 4, only has a trivial solution, as shown in the following theorem.

Theorem 0.6.2 *Consider Case 4. Problem (18) admits an* *optimal solution if and only if $y_0 = 0$, in which case the unique optimal solution is $Y^* = 0$.*

Example 0.6.1 Let $f(x) = x_-^2$. This is the mean–semivariance model investigated in Section 0.4. It then follows from Theorem 0.6.1 that the mean–semivariance model admits an optimal solution if and only if $z = x_0/E\rho$.

Example 0.6.2 Let $f(x) = |x|$. In view of Theorem 0.6.2 the continuous-time mean–absolute-deviation model admits an optimal solution if and only if $z = x_0/E\rho$, in which case the optimal portfolio is simply the risk-free one.

Example 0.6.3 Let $f(x) = e^{-x}$. By Theorem 0.6.1, the mean–risk portfolio selection problem admits an optimal solution if and only if $x_0 - zE\rho \in [(E\rho)(E\ln\rho) - E(\rho\ln\rho), 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0-(E\rho)(E\ln\rho)+E(\rho\ln\rho)}{E\rho}]$.

0.7 Concluding Remarks

We have solved out the Mean-Weighted-variance problem in a complete continuous-time financial market, and proved that other than a trivial case, Mean-Semivariance problem is not well-posed. Furthermore, for a kind of downside risk, we proved that the corresponding Mean-Downside-risk problem is also not well-posed other than a trivial case.

Not all the results in this paper are negative. We turned to

Mean-Risk problem, where the risk is measure by the expectation of a (strictly) convex function of the terminal wealth. We got a sufficient and necessary condition for the Mean-Risk problem to be well-posed.

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